On the Dimension of the Quotient Ring R/K Where K is a Complement

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Abstract

The aim of the present paper is to obtain some interesting results related to the concept "finite dimension" in the theory of associative rings R with respect to two sided ideals. It is known that if an ideal H of R has finite dimension, then there exist uniform ideals U_i, $1 \le i \le n$ of R such that the sum $U_1 \oplus U_2 \oplus \dots \oplus U_n$ is essential in H. This n is independent of choice of uniform ideals and we call it as dimension of H (we write dim H, in short). We obtain some important relations between the concepts complement ideals and essential ideals. Finally, we proved that dim(R/K) = dim R – dim K for a complement ideal K of R. We include some necessary examples.

Keywords: Ring, Two Sided Ideal, Essential Ideal, Uniform Ideal, Finite Dimension, Complement Ideal.

1. INTRODUCTION

The dimension of a vector space is defined as the number of elements in the basis. One can define a basis of a vector space as a maximal set of linearly independent vectors or a minimal set of vectors which span the space. The former when generalized to modules over rings becomes the concept of Goldie dimension. Goldie proved a structure theorem for modules which states that "a module with finite Goldie dimension (FGD, in short) contains a finite number of uniform submodules $U_1, U_2, \dots, U_n$ whose sum is direct and essential in M". The number n obtained here is independent of the choice of $U_1, U_2, \dots, U_n$ and it is called as Goldie dimension of M. The concept Goldie dimension in Modules was studied by several authors like Satyanarayana, Mohiddin Shaw.

If we consider ring as a module over itself, then the existing literature tells about dimension theory for ideals (i.e., two sided ideals) in case of commutative rings; and left (or right) ideals in case of associative (but not commutative) rings. So we can understand the structure theorem for associative rings in terms of one sided ideals only (that is, if R has FGD with respect to left (right) ideals, then there exist n uniform left (or right) ideals of R whose sum is direct and essential in R). This result cannot say about the structure theorem for associative rings in terms of two sided ideals.

To fill this gap, Satyanarayana, Nagaraju, Balamurugan & Godloza [4] started studying the concepts: complement, essential, uniform, finite dimension with respect to two sided ideals of R. We say a ring R has *finite dimension on ideals* (FDI, in short) if R does not contain an infinite number of non-zero ideals of R whose sum is direct. A non- zero ideal K of R is said to have *finite dimension on ideals of R* (FDIR, in short) if K does not contain an infinite number of non-zero ideals of R whose sum is direct. It is clear that if R has FDI, then every non-zero ideal of R has FDIR.

Now we state some definitions and results from [4 & 5] that are useful in the later part of this paper. We write "$I \trianglelefteq R$" to denote "I is an ideal (two sided ideal) of R".

1.1 Definitions: Let $I \trianglelefteq R$, $J \trianglelefteq R$ such that $I \subseteq J$.

(i). We say that I is *essential* (or *ideal essential*) in J if it satisfies the following condition: $K \trianglelefteq R$, $K \subseteq J$, $I \cap K = (0)$ imply $K = (0)$.
(ii). If I is essential in J and $I \neq J$, then we say that J is a *proper essential extension* of I. If I is essential in J, then we denote this fact by $I \leq_e J$.

(iii). If $K \trianglelefteq R$, $A \trianglelefteq R$ and K is a maximal element in $\{I / I \trianglelefteq R, \ I \cap A = (0)\}$, then we say that K is a *complement* of A (or a *complement* in R).

1.2 Note: If A, B, C are ideals of R, $A \subseteq C$, $A \cap B = (0)$ and C is a complement of B, then $C \oplus B \leq_e R$, and C is an essential extension of A.

1.3 Result (2.4 of [4]): (i) If $I \trianglelefteq R$, $J \trianglelefteq R$, $K \trianglelefteq R$ such that $I \leq_e J$, and $J \leq_e K$, then $I \leq_e K$;
(ii) If $I \subseteq J \subseteq K$, then $I \leq_e K$ if and only if $I \leq_e J$, and $J \leq_e K$; and
(iii) If R, S are two rings, $f: R \to S$ is a ring isomorphism, and A is an ideal of R, then $A \leq_e R \Leftrightarrow f(A) \leq_e S$.

1.4 Lemma (2.7 of [4]): Let K_1, K_2, … K_t, L_1, L_2, … L_t are ideals of R such that the sum $K_1 + K_2 + \ldots + K_t$ is direct and $L_i \subseteq K_i$ for $1 \leq i \leq t$. Then $L_1 + L_2 + \ldots + L_t \leq_e K_1 + K_2 + \ldots + K_t \Leftrightarrow L_i \leq_e K_i$ for $1 \leq i \leq t$.

1.5 Definition: A non-zero ideal I of R is said to be *uniform* if $(0) \neq J \trianglelefteq R$, and $J \subseteq I \Rightarrow J \leq_e I$.

1.6 Theorem (3.3 of [4]): (i) I is an uniform ideal $\Leftrightarrow L \trianglelefteq R$, $K \trianglelefteq R$, $L \subseteq I$, $K \subseteq I$, $L \cap K = (0) \Rightarrow L = (0)$ or $K = (0)$.

(ii) Let R and S be two rings and $f: R \to S$ be ring isomorphism. If $U \trianglelefteq R$, then U is uniform in R $\Leftrightarrow$ f(U) is uniform in S.
(iii) If U and K are two ideals of R such that $U \cap K = (0)$, then U is uniform in R $\Leftrightarrow$ (U + K)/K is uniform in R/K.

(iv). If R has FDI and $(0) \neq K \trianglelefteq R$, then K contains an uniform ideal of R.

Now we state the main theorem on [4].

1.7 Theorem (4.4 of [4]): Suppose $0 \neq H \trianglelefteq R$ and H has FDIR. Then the following conditions hold.
(i) (Existence) There exist uniform ideals $U_1, U_2, \ldots U_n$ of R whose sum is direct and essential in H;
(ii) (Uniqueness) If V_i, $1 \leq i \leq k$ are uniform ideals of R whose sum is direct and essential in H, then $k = n$.

The number is independent of the choice of the uniform ideals U_i, $1 \leq i \leq n$. This number n is called the *dimension* of H and it is denoted by dim H.

1.8 Theorem (2.2 of [5]): Suppose R has FDI.

(i). If $H \trianglelefteq R$, $K \trianglelefteq R$ and $H \subseteq K$, then $\dim H \leq \dim K$;
(ii) If $(0) \neq A_i$ is an ideal of R for all i, $1 \leq i \leq t$ whose sum is direct, and $A_i \subseteq H$, $1 \leq i \leq t$, then $\dim H \geq t$;
(iii) H is uniform $\Leftrightarrow \dim H = 1$;
(iv) If H is a non-zero ideal of R, then $\dim H \geq 1$;
(v) If I_i, $1 \leq i \leq k$ are uniform ideals of R whose sum is direct, then $k \leq \dim R$. Moreover $\dim H = \max\{k$ / there exist uniform ideals I_i, $1 \leq i \leq k$ of R whose sum is direct, $I_i \subseteq H$, $1 \leq i \leq k\}$;
(vi). If $n = \dim R$, then the number of summands in any decomposition of a given ideal I of R as a direct sum of non-zero ideals of R is at most n.; and
(vii) If $f: R \rightarrow S$ is an isomorphism and R has FDI, then S has FDI and $\dim R = \dim S$.

1.9 Result (2.3 of [5]): If H and K are ideals of R with $H \cap K = (0)$, then $\dim (K + H) = \dim K + \dim H$.

1.10 Theorem (3.1 of [5]): If R has FDI with $\dim R = n$ and $H \trianglelefteq R$, then the following conditions are equivalent:
(i). $H \leq_e R$; (ii). $\dim H = \dim R$; and (iii). H contains a direct sum of n uniform ideals.

1.11 Proposition (3.4 of [5]): Suppose R has FDI and $K \trianglelefteq R$.
(i) K is a complement ideal $\Leftrightarrow$ K has no proper essential extensions; and
(ii) If K is a complement, then R/K has FDI, and $\dim(R/K) \leq \dim R$.

The aim of the present paper is to continue the study of rings with FDI. Section-2, deals with the concepts: Complement and Essential Ideals. In Section-3, we include an example of an ideal K of R with $\dim R/K \neq \dim R - \dim K$. Finally, we proved that if K is a complement ideal of R, then $\dim R/K = \dim R - \dim K$.

Throughout this paper R stands for a fixed (not necessarily commutative) ring with FDI.

2. COMPLEMENT AND ESSENTIAL IDEALS

2.1 Lemma: Let K be an ideal of R and $\pi: R \rightarrow R/K$ be the canonical epimorphism. Then the following two conditions are equivalent:
(i) K is a complement; and
(ii) For any ideal K^1 of R containing K, we have that K^1 is a complement in R if and only if $\pi(K^1)$ is complement in R/K.

Proof: *(i) $\Rightarrow$ (ii)*: Suppose that K is a complement of an ideal Z of R. Suppose K^1 is a complement ideal of R containing K. Now K^1 is a complement of some ideal S of R. To show $\pi(K^1)$ is a complement of $\pi(S)$, it is enough to verify that $\pi(K^1)$ is maximal with respect to the property $\pi(K^1) \cap \pi(S) = (0)$.

Let $x \in \pi(K^1) \cap \pi(S) \Rightarrow x \in \pi(K^1)$ and $x \in \pi(S) \Rightarrow x = k^1 + K$ and $x = s + K$, for some $k^1 \in K^1$ and $s \in S \Rightarrow s - k^1 \in K \subseteq K^1 \Rightarrow s \in K^1$ (since $k^1 \in K^1$) $\Rightarrow s \in K^1 \cap S$ (since $s \in S$) $\Rightarrow s = 0$ (since $K^1 \cap S = (0)$) $\Rightarrow x = s + K = 0$. Therefore $\pi(K^1) \cap \pi(S) = (0)$.

Let A be an ideal of R/K such that $A \supsetneq \pi(K^1)$. It is obvious that $A = \pi(K^*)$ for some ideal K^* of R with $K^* \supseteq K^1$. If $K^* = K^1$, then $A = \pi(K^*) = \pi(K^1)$, a contradiction. So $K^* \supsetneq K^1$. Since K^1 is a complement of S, we have that $K^* \cap S \neq (0)$. Let $0 \neq y \in K^* \cap S$. Now $y + K \in \pi(K^*) \cap \pi(S)$. If $y + K = 0$, then $y \in K \Rightarrow y \in K \cap S \subseteq K^* \cap S = (0)$, a contradiction. Hence $0 \neq y + K \in \pi(K^*) \cap \pi(S)$. This shows that $\pi(K^1)$ is a complement of $\pi(S)$.

Conversely suppose that $\pi(K^1) = K^1/K$ is a complement of an ideal $\pi(I) = I/K$ of R/K. Now we have to verify that K^1 is a complement ideal in R. By Note 1.2, there exists a complement X of K such that $Z \subseteq X$. Since $K^1 \cap I = K$, we have that $K^1 \cap (I \cap Z) = (K^1 \cap I) \cap Z = K \cap Z = (0)$ and so $K^1 \cap (I \cap Z) = (0)$. Let Y be a complement of $I \cap Z$ with $Y \supseteq K^1$. Now since $Y \supseteq K^1 \supseteq K$ and $I \supseteq K$, we have $Y \cap I \supseteq K$. Also $(Y \cap I) \cap Z = Y \cap (I \cap Z) = (0)$. Since $Y \cap I \supseteq K$, $(Y \cap I) \cap Z = (0)$ and K is a complement of Z, it follows that $Y \cap I = K$. So $\pi(Y) \cap \pi(I) = (0)$. Since $Y \supseteq K^1$, we have $\pi(Y) \supseteq \pi(K^1)$. Now $\pi(Y) \supseteq \pi(K^1)$, $\pi(Y) \cap \pi(I) = (0)$ and $\pi(K^1)$ is a complement of $\pi(I)$, it follows that $\pi(Y) = \pi(K^1)$. Now we have that $Y = K^1$. [*Verification*: We know that $Y \supseteq K^1$. Let $x \in Y$. Then $\pi(x) \in \pi(Y) = \pi(K^1) \Rightarrow x + K \in \pi(K^1) \Rightarrow x + K = y + K$, for some $y \in K^1$ $\Rightarrow x - y \in K \subseteq K^1$ and $y \in K^1 \Rightarrow x - y \in K^1$ and $y \in K^1 \Rightarrow x \in K^1$. Therefore $Y = K^1$]. Since Y is a complement, we conclude that K^1 is a complement.

(ii) ⇒ (i): Since K is an ideal of R containing K, and since $\pi(K) = 0$ is a complement in R/K, it follows that K is complement in R.

2.2 Lemma: Let $K \trianglelefteq R$ and $\pi: R \to R/K$ be the canonical epimorphism. Then the following two conditions are equivalent:
(i) K is a complement; and
(ii) For any essential ideal S of R, $\pi(S)$ is essential in R/K.

Proof: *(i) ⇒ (ii)*: Let S be an essential ideal of R. To show $\pi(S)$ is essential in R/K, take an ideal Z/K of R/K such that $\pi(S) \cap (Z/K) = (0)$. It is enough to show $Z = K$. In a contrary way, suppose $Z \neq K$. Then by Proposition 1.11 (i), K has no proper essential extensions. So K is not essential in Z and hence there exists an ideal $(0) \neq A$ of R such that $A \cap K = (0)$ and $A \subseteq Z$. Since S is essential in R, there exists $0 \neq x \in S \cap A \Rightarrow \pi(x) \in \pi(S) \cap \pi(A) \subseteq \pi(S) \cap \pi(Z) = (0)$ $\Rightarrow \pi(x) = 0 \Rightarrow x + K = 0 \Rightarrow x \in K \Rightarrow x \in K \cap A$ (since $x \in A$) $= (0) \Rightarrow x = 0$, a contradiction. Thus $Z = K$. We proved that $\pi(S)$ is essential in R/K.

(ii) ⇒ (i): Assume the converse hypothesis. In a contrary way, suppose that K is not a complement. By Proposition 1.11 (i), K has a proper essential extension K^*. Let X be a complement of K^* in R. Then $K^* \oplus X$ is essential in R (by Note 1.2). Since K is essential in K^* by Lemma 1.4, $K \oplus X$ is essential in $K^* \oplus X$ and so $K \oplus X$ is essential in R (by Result 1.3 (i)). By the converse hypothesis, we get that $\pi(K + X)$ is essential in R/K. Since K^* contains K properly, $\pi(K^*)$ is a non-zero ideal of R/K. Now $(K + X) \cap K^* = K + (X \cap K^*) = K$, which shows that $\pi(K + X) \cap \pi(K^*) = (0)$. This is a contradiction to the fact that $\pi(K + X)$ is essential in R/K.

Combining Lemmas 2.1 and 2.2, we get the following Theorem.

2.3 Theorem: Let K be an ideal of R and $\pi: R \to R/K$ be the canonical epimorphism. Then the following three conditions are equivalent:
(i) K is a complement;
(ii) For any ideal K^1 of R containing K, we have that K^1 is a complement in R if and only if $\pi(K^1)$ is complement in R/K; and
(iii) For any essential ideal S of R, $\pi(S)$ is essential in R/K.

3. DIMENSION OF THE QUOTIENT RING R/K

3.1 Lemma: Let R be a Ring with FDI. If A is an ideal of R such that dim(R/A) = 1 and A is not essential in R, then dim(R/A) = dim R - dim A.

Proof: Since A is not essential, there is a non-zero ideal I of R such that $A \cap I = (0)$. Let K be a complement of A containing I. Suppose dim K $\geq$ 2. Then K contains a direct sum of two uniform ideals I_1 and I_2 of R. Clearly $I_i \cap A = (0)$ for i = 1, 2. By Theorem 1.6 (iii), $\left(\frac{I_1 + A}{A}\right)$, $\left(\frac{I_2 + A}{A}\right)$ are two uniform ideals of R/A. It is easy to verify that the sum $\left(\frac{I_1 + A}{A}\right) + \left(\frac{I_2 + A}{A}\right)$ is direct and hence dim(R/A) $\geq$ 2, a contradiction. Hence dim K $\not\geq$ 2. Since K $\neq$ (0), by Theorem 1.8 (iv), we have that dim K $\geq$ 1. Therefore dim K = 1. Since K is complement of A, the sum K + A is direct and essential in R. So dim R = dim(K + A) (by Theorem 1.10) = dim K + dim A (by Result 1.9) = 1 + dim A = dim (R/A) + dim A. Hence dim(R/A) = dim R - dim A.

It is well known that if V is a finite dimensional vector space and W is a subspace of V, then dim(V/W) = dim V - dim W. This dimension condition may not hold for a general ideal W of a Ring V where "dim" denotes the "finite dimension". For this, observe the following examples.

3.2 Examples: Write R = $\mathbb{Z}$, the ring of integers. Since every ideal of $\mathbb{Z}$ is essential in $\mathbb{Z}$, it follows that $\mathbb{Z}$ is uniform and so dim R = 1.

(i) Write K = $6\mathbb{Z}$. Now K is an uniform ideal of R. So dim K = 1 and dim R - dim K = 1 - 1 = 0. Now $R/K = \mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}_6 \cong \mathbb{Z}_2 + \mathbb{Z}_3$ and so dim(R/K) = 2.
Thus dim(R/K) = 2 $\neq$ 0 = dim R - dim K.

(ii) Let p, q be distinct primes and consider H, the ideal of $\mathbb{Z}$ generated by the product of these primes (that is, H = $pq\mathbb{Z}$). Now H is uniform ideal and so dim H = 1. It is known that $\mathbb{Z}/H = \mathbb{Z}_{pq} \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$, and $\mathbb{Z}_p$, $\mathbb{Z}_q$ are uniform ideals. So dim($\mathbb{Z}$/H) = 2. Thus dim ($\mathbb{Z}$/H) = 2 $\neq$ 0 = 1 - 1 = dim $\mathbb{Z}$ - dim H.

Hence, there arise a type of ideals K which satisfy the condition dim(R/K) = dim R – dim K.

3.3 Theorem: If R has FDI and K is a complement ideal, then dim(R/K) = dim R – dim K.

Proof: By Proposition 1.11 (ii), we have that R/K has FDI. If dim(R/K) = 1, then by Lemma 3.1, dim(R/K) = dim R – dim K. Suppose dim(R/K) = m, where m $\geq$ 2. Then by Theorem 1.7, there exist ideals $K_1, K_2, \ldots, K_m$ of R containing K properly such that K_i/K is an uniform ideal for $1 \leq i \leq m$, the sum $(K_1/K) + (K_2/K) + \ldots + (K_m/K)$ is direct and essential in R/K. Clearly $K = K_i \cap K_j$, $i \neq j$ for $1 \leq i \leq m$ and $1 \leq j \leq m$. Since K is a complement ideal of R, by Proposition 1.11 (i), we have that K is not essential in K_i, $1 \leq i \leq m$. So there exist uniform ideals I_i ($1 \leq i \leq m$) of R such that $I_i \subseteq K_i$ and $I_i \cap K = (0)$. By a straight forward verification, we get that the sum $K + I_1 + I_2 + \ldots + I_m$ is direct. Now we verify that $T = K + I_1 + I_2 + \ldots + I_m$ is essential in R. Let H be an ideal of R such that $T \cap H = (0)$. Then $T \cap (H + K) = (T \cap H) + K$ (by modular law) = (0) + K = K. So $(T/K) \cap \frac{(H+K)}{K} = (0)$. Since $\frac{(I_i+K)}{K}$ is a non-zero ideal of the uniform ideal K_i/K, it follows that $\frac{(I_i+K)}{K}$ is essential in K_i/K, for $1 \leq i \leq m$. By Lemma 1.4, $\frac{(I_1+K)}{K} + \frac{(I_2+K)}{K} + \ldots + \frac{(I_m+K)}{K}$ is essential in R/K. Therefore T/K is essential in R/K. Since $(T/K) \cap \frac{(H+K)}{K} = (0)$, we have

$\frac{(H+K)}{K}$ = (0) and so $H \subseteq K$. So $H = H \cap K \subseteq H \cap T = (0)$. This shows that the sum $T = K + I_1 + \ldots + I_m$ is essential in R. Now dim R = dim($K + I_1 + \ldots + I_m$) (by Theorem 1.10) = dim K + dim I_1 + dim I_2 + … + I_m (by Result 1.9) = $\dim K + \underbrace{1+\ldots+1}_{\text{m-terms}}$ (by Theorem 1.8 (iii)) = dim K + m = dim K + dim(R/K). Therefore dim(R/K) = dim R – dim K.

4. COMPARISON WITH THE PREVIOUS WORK DONE IN THE RELEVANT FIELDS

(i) A module is a generalized concept of vector space. If W is a subspace of a vector space V and dim W = dim V, then V = W. But in case of modules, if W is a submodule of a module M with dim W = dim M, then W is essential in M, but W may not be equal to M. Due to this fact the study of Goldie dimension in modules becomes important. A ring is a module over itself. So the theory developed in modules is also a contribution to the theory of Rings.

(ii) A ring R is a module R_R. The right ideals in R coincide with the submodules of R_R. So the dimension theory developed in modules speaks about the results related to the dimension of one sided "right ideals" of rings. But the results obtained in module theory can not speak about the dimension of two sided ideals of rings. To fill this gap, Satyanarayana, Nagaraju, Bala Murugan, and Godloza [4] started studying the concept 'dimension of two sided ideals' in rings. The study was continued in [5] and [7]. The further study on this concept formed the results of the present paper.'

5. CONCLUSION & FUTURE WORK

This paper is the continuation of the published papers [4] and [5]. In this present paper, we are able to obtain several interesting results related to the concept dimension of rings with respect to two sided ideals. We proved fundamental and critical relations between complement ideals and essential ideals. In general the statement:
dim(R/K) = dim R – dim K, is not true for two sided ideals K. To explain this fact, an example was presented. Finally we achieved the result and able to prove the important statement that dim(R/K) = dim R – dim K, for a particular type of submodule (namely, complement submodule). We continue this work, in near future, to get some more important dimension conditions in rings with respect to two sided ideals.

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