E-Cordial Labeling of Some Mirror Graphs

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Abstract

Let $G$ be a bipartite graph with a partite sets $V_1$ and $V_2$ and $G'$ be the copy of $G$ with corresponding partite sets $V_1'$ and $V_2'$. The mirror graph $M(G)$ of $G$ is obtained from $G$ and $G'$ by joining each vertex of $V_1$ to its corresponding vertex in $V_2$ by an edge. Here we investigate E-cordial labeling of some mirror graphs. We prove that the mirror graphs of even cycle $C_n$, even path $P_n$ and hypercube $Q_k$ are E-cordial graphs.

Keywords: E-Cordial labeling, Edge graceful labeling, Mirror graphs.

AMS Subject Classification Number(2010): 05C78

1. INTRODUCTION

We begin with finite, connected and undirected graph $G = (V(G), E(G))$ without loops and multiple edges. For standard terminology and notations we refer to West[1]. The brief summary of definitions and relevant results are given below.

Definition 1.1

If the vertices of the graph are assigned values subject to certain condition(s) then it is known as graph labeling.

Most of the graph labeling techniques trace their origin to graceful labeling introduced independently by Rosa[3] and Golomb[4] which is defined as follows.

Definition 1.2

A function $f$ is called graceful labeling of graph $G$ if $f : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ is injective and the induced function $f^*(e = uv) = f(u) - f(v)$ is bijective. A graph which admits graceful labeling is called a graceful graph.


Definition 1.3

A graph $G$ is said to be edge-graceful if there exists a bijection $f : E(G) \rightarrow \{1, 2, \ldots, |E|\}$ such that the induced mapping $f^* : V(G) \rightarrow \{0, 1, 2, \ldots, |V| - 1\}$ given by $f^*(x) = \sum_{xy \in E(G)} f(xy)(mod |V|)$, $xy \in E(G)$.

Definition 1.4

A mapping $f : V(G) \rightarrow \{0, 1\}$ is called binary vertex labeling of $G$ and $f(v)$ is called the label of vertex $v$ of $G$ under $f$.
Notations 1.5
For an edge $e = uv$, the induced edge labeling $f^* : E(G) \to \{0,1\}$ is given by $f^*(e = uv) = f(u) - f(v) \mod 2$. Let $v_f(0), v_f(1)$ be the number of vertices of $G$ having labels 0 and 1 respectively under $f$ and let $e_f(0), e_f(1)$ be the number of edges of $G$ having labels 0 and 1 respectively under $f^*$.

Definition 1.6
A binary vertex labeling of graph $G$ is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph $G$ is called cordial if admits cordial labeling.

The concept of cordial labeling was introduced by Cahit[6]. He also investigated several results on this newly introduced concept.

Definition 1.7
Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$ and let $f : E(G) \to \{0,1\}$. Define $f$ on $V(G)$ by $f(v) = \sum_{uv \in E(G)} f(uv) \mod 2$. The function $f$ is called an E-cordial labeling of $G$ if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph is called E-cordial if admits E-cordial labeling.

In 1997 Yilmaz and Cahit[7] introduced E-cordial labeling as a weaker version of edge-graceful labeling and having flavour of cordial labeling. They proved that the trees with $n$ vertices, $K_n$, $C_n$ are E-cordial if and only if $2(4n + 4) \equiv 0 \mod 4$ while $K_m,n$ admits E-cordial labeling if and only if $m + n \equiv 2 \mod 4$.

Definition 1.8
For a bipartite graph $G$ with partite sets $V_1$ and $V_2$. Let $G'$ be the copy of $G$ and $V'_1$ and $V'_2$ be the copies of $V_1$ and $V_2$. The mirror graph $M(G)$ of $G$ is obtained from $G$ and $G'$ by joining each vertex of $V_2$ to its corresponding vertex in $V'_2$ by an edge.

Lee and Liu[8] have introduced mirror graph during the discussion of $k$ - graceful labeling. Devaraj[9] has shown that $M(m,n)$, the mirror graph of $K_{m,n}$ is E-cordial when $m + n$ is even while the generalized Petersen graph $P(n,k)$ is E-cordial when $n$ is even.

In the following section we have investigated some new results on E-cordial labeling for some mirror graphs.

2. Main Results
Theorem 2.1 Mirror graph of even cycle $C_n$ is E-cordial.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_n$ be the edges of cycle $C_n$, where $n$ is even and $G = C_n$. Let $V_1 = \{v_2, v_3, \ldots, v_n\}$ and $V_2 = \{v_1, v_3, \ldots, v_{n-1}\}$ be the partite sets of $C_n$. Let $G'$ be the copy of $G$ and $V'_1 = \{v'_2, v'_3, \ldots, v'_n\}$ and $V'_2 = \{v'_1, v'_3, \ldots, v'_{n-1}\}$ be the copies of $V_1$ and $V_2$ respectively. Let $e'_1, e'_2, \ldots, e'_n$ be the edges of $G'$. The mirror graph $M(G)$ of $G$ is obtained from $G$ and $G'$ by joining each vertex of $V_2$ to its corresponding vertex in $V'_2$ by additional edges $e'_1, e'_2, \ldots, e'_n$.

We note that $|V(M(G))| = 2n$ and $|E(M(G))| = 2n + \frac{n}{2}$. Let $f : E(M(G)) \to \{0,1\}$ as follows:

For $1 \leq i \leq n$:
$$f(e_i) = 1; \quad i \equiv 0, 1 \pmod{4}.$$ 
$$f(e_i) = 0; \quad otherwise.$$ 

$$f(e'_i) = 1; \quad i \equiv 1, 2 \pmod{4}.$$ 
$$f(e'_i) = 0; \quad otherwise.$$ 

For $1 \leq j < \frac{n}{2}$:

$$f(e'_j) = 1; \quad j \equiv 1 \pmod{2}.$$ 
$$0; \quad otherwise.$$ 

For $j = \frac{n}{2}$:

$$f(e'_j) = 0.$$ 

In view of the above defined labeling pattern $f$ satisfies conditions for E-cordial labeling as shown in Table 1. That is, the mirror graph of even cycle $C_n$ is E-cordial.

<table>
<thead>
<tr>
<th>vertex condition</th>
<th>edge condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \equiv 0 \pmod{4}$</td>
<td>$v_j(0) = v_{j+1}(1) = n$</td>
</tr>
<tr>
<td>$n \equiv 2 \pmod{4}$</td>
<td>$v_j(0) = v_{j+1}(1) = n$</td>
</tr>
</tbody>
</table>

Table 1

Illustration 2.2: The E-cordial labeling for the mirror graph of cycle $C_6$ is shown in Figure 1.

Theorem 2.3: Mirror graph of path $P_n$ is E-cordial for even $n$.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_{n-1}$ be the edges of path $P_n$ where $n$ is even and $G = P_n$. $P_n$ is a bipartite graph. Let $V'_1 = \{v_2, v_4, \ldots, v_n\}$ and $V'_2 = \{v_1, v_3, \ldots, v_{n-1}\}$ be the bipartition of $P_n$. Let $G'$ be a copy of $G$ and $V'_1 = \{v'_2, v'_4, \ldots, v'_n\}$ and $V'_2 = \{v'_1, v'_3, \ldots, v'_{n-1}\}$ be the copies of $V'_1$ and $V'_2$. Let $e'_1, e'_2, \ldots, e'_{n-1}$ be the edges of $G'$. The mirror graph $M(G)$ of $G$ is obtained from $G$ and $G'$ by
joining each vertex of \( V_2 \) to its corresponding vertex in \( V'_2 \) by additional edges \( e'_1, e'_2, \ldots, e'_n \).

We note that \( |V(M(G))| = 2n \) and \( |E(M(G))| = 2(n-1) + \frac{n}{2} \). Let \( f : E(M(G)) \to \{0,1\} \) as follows:

For \( 1 \leq i < n-1 \):

\[
f(e_i) = \begin{cases} 1; & i \equiv 0,1 \pmod{4} \\ 0; & \text{otherwise}\end{cases}
\]

For \( i = n-1 \):

\[
f(e_{n-1}) = 1.
\]

For \( 1 \leq i \leq n-1 \):

\[
f(e'_i) = \begin{cases} 1; & i \equiv 0,3 \pmod{4} \\ 0; & \text{otherwise}\end{cases}
\]

For \( 1 \leq j \leq \frac{n}{2} \):

\[
f(e'_j) = \begin{cases} 1; & j \equiv 0 \pmod{2} \\ 0; & \text{otherwise}\end{cases}
\]

In view of the above defined labeling pattern \( f \) satisfies the conditions for E-cordial labeling as shown in Table 2. That is, the mirror graph of path \( P_n \) is E-cordial for even \( n \).

<table>
<thead>
<tr>
<th>vertex condition</th>
<th>edge condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n \equiv 0 \pmod{4} )</td>
<td>( v_j(0) = v_j(1) = n )</td>
</tr>
<tr>
<td>( n \equiv 2 \pmod{4} )</td>
<td>( v_j(0) = v_j(1) = n )</td>
</tr>
</tbody>
</table>

**Illustration 2.4:** The E-cordial labeling for mirror graph of path \( P_8 \) is shown in Figure 2.

![Figure 2](image-url)
Theorem 2.5: Mirror graph of hypercube $Q_k$ is E-cordial.

Proof:
Let $G = Q_k$ be a hypercube with $n$ vertices where $n = 2^k$. Let $V_1$ and $V_2$ be the bipartition of $Q_k$ and $G'$ be a copy of $G$ with $V_1'$ and $V_2'$ be the copies of $V_1$ and $V_2$ respectively. Let $e_1, e_2, \ldots, e_m$ be the edges of graph $G$ and $e_1', e_2', \ldots, e_m'$ be the edges of graph $G'$ where $m = \frac{nk}{2}$. The mirror graph $M(G)$ of $G$ is obtained from $G$ and $G'$ by joining each vertex of $V_2$ to its corresponding vertex in $V_2'$ by additional edges $e_1', e_2', \ldots, e_m'$ then $|V(M(G))| = 2n$ and $|E(M(G))| = \frac{n(2k + 1)}{2}$.

Define $f : E(M(G)) \rightarrow \{0, 1\}$ as follows:

Case 1: $k \equiv 0 \pmod{2}$

Let $V_i = \{v_{i1}, v_{i2}, \ldots, v_{in}\}$ and $V'_i = \{v'_{i1}, v'_{i2}, \ldots, v'_{in}\}$ where $i = 1, 2$. All the edges incident to the vertices $v_{ij}$ and $v'_{ij}$ where $j \equiv 1 \pmod{2}$ are assigned the label 0 while the edges incident to the vertices $v_{ij}$ and $v'_{ij}$ where $j \equiv 0 \pmod{2}$ are assigned label 1.

For $1 \leq j \leq \frac{n}{2}$:

- $f(e_j') = 1$; $j \equiv 0 \pmod{2}$.
- $f(e_j') = 0$; otherwise.

Case 2: $k \equiv 1 \pmod{2}$

For $1 \leq i \leq n$:

- $f(e_i) = 1$.

For $1 \leq i \leq n$:

- $f(e_i') = 0$.

For $1 \leq j \leq \frac{n}{2}$:

- $f(e_j') = 1$; $j \equiv 0 \pmod{2}$.
- $f(e_j') = 0$; otherwise.

In view of the above defined labeling pattern $f$ satisfies the conditions for E-cordial labeling as shown in Table 3.

<table>
<thead>
<tr>
<th>vertex condition</th>
<th>edge condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$v_j(0) = v_j(1) = n$</td>
</tr>
</tbody>
</table>

That is, the mirror graph of hypercube $Q_k$ is E-cordial.

Illustration: 2.6

The E-cordial labeling for mirror graph of hypercube $Q_3$ is shown in Figure 3.
3. CONCLUDING REMARKS
Here we investigate E-cordial labeling for some mirror graphs. To investigate similar results for other graph families and in the context of different graph labeling problems is an open area of research.

REFERENCES