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The initial efforts helped to shape the editorial policy and to sharpen the focus of the journal. Started with Volume 8, 2014, SPIJ appears with more focused issues related to signal processing studies. Besides normal publications, SPIJ intend to organized special issues on more focused topics. Each special issue will have a designated editor (editors) – either member of the editorial board or another recognized specialist in the respective field.

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Image Denoising Based On Sparse Representation In A Probabilistic Framework

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Abstract

Image denoising is an interesting inverse problem. By denoising we mean finding a clean image, given a noisy one. In this paper, we propose a novel image denoising technique based on the generalized k density model as an extension to the probabilistic framework for solving image denoising problem. The approach is based on using overcomplete basis dictionary for sparsely representing the image under interest. To learn the overcomplete basis, we used the generalized k density model based ICA. The learned dictionary used after that for denoising speech signals and other images. Experimental results confirm the effectiveness of the proposed method for image denoising. The comparison with other denoising methods is also made and it is shown that the proposed method produces the best denoising effect.

Keywords: Sparse Representation, Image Denosing, Independent Component Analysis, Dictionary Learning.

1. INTRODUCTION

Being a simple inverse problem, the denoising is a challenging task and basically addresses the problem of estimating a signal from the noisy measured version available from that. A very common assumption is that the present noise is additive zero-mean white Gaussian with standard deviation σ . In this paper, we only consider the contaminated source, noise, of natural images. In other words, the purpose of image denoising is to restore the original image with noise-free. This problem appears to be very simple however that is not so when considered under practical situations, where the type of noise, amount of noise and the type of images all are variable parameters, and the single algorithm or approach can never be sufficient to achieve satisfactory results.

Many solutions have been proposed for this problem based on different ideas, such as statistical modeling [1], spatial adaptive filters, diffusion enhancement [2], transfer domain methods [3,4], order statistics [5], independent component analysis (ICA) and standard sparse coding (SC) shrinkage proposed by Alpo Hyvärinen in 1997 [6,7], and yet many more. Among these methods, methods based on sparse and redundant representations has recently attracted lots of attentions [8]. Many researchers have reported that such representations are highly effective and promising toward this stated problem [8]. Sparse representations firstly examined with unitary wavelet dictionaries leading to the well-known shrinkage algorithm [5]. A major motivation of using overcomplete representations is mainly to obtain translation invariant property [9]. In this respect, several multiresolutional and directional redundant transforms are introduced and applied to denoising such as curvelets, contourlets, wedgelets, bandlets and the steerable wavelet [5,8].

Moreover, the Ref. [10] gave an important conclusion: when ICA is applied to natural image data, ICA is equivalent to SC. However, ICA emphasizes independence over sparsity in the output coefficients, while SC requires that the output coefficients must be sparse and as independent as possible. Because of the sparse structures of natural images, SC is more suitable to process natural images than ICA. Hence, SC method has been widely used in natural image processing [10,11].

The now popular sparse signal models, on the other hand, assume that the signals can be accurately represented with a few coefficients selecting atoms in some dictionary[12]. Recently, very impressive image restoration results have been obtained with local patch-based sparse representations calculated with dictionaries learned from natural images [13,14]. Relative to pre-fixed dictionaries such as wavelets [1], curve lets [15], and band lets [16], learned dictionaries enjoy the advantage of being better adapted to the images, thereby enhancing the sparsity.

However, dictionary learning is a large-scale and highly non-convex problem. It requires high computational complexity, and its mathematical behavior is not yet well understood. In the dictionaries aforementioned, the actual sparse image representation is calculated with relatively expensive non-linear estimations. Such as ℓ_1 or matching pursuits [17,18]. More importantly, as will be reviewed, with a full degree of freedom in selecting the approximation space (atoms of the dictionary), non-linear sparse inverse problem estimation may be unstable and imprecise due to the coherence of the dictionary [19].

Structured sparse image representation models further regularize the sparse estimation by assuming de- pendency on the selection of the active atoms. One simultaneously selects blocks of approximation atoms, thereby reducing the number of possible approximation spaces [20,21]. These structured approximations have been shown to improve the signal estimation in a compressive sensing context for a random operator. However, for more unstable inverse problems such as zooming or deblurring, this regularization by itself is not sufficient to reach state-of-the-art results. Recently some good image zooming results have been obtained with structured sparsity based on directional block structures in wavelet representations [19]. However, this directional regularization is not general enough to be extended to solve other inverse problems.

In this paper we show that the over complete basis dictionary which learning by using the ICA probabilistic technique can capture the main structure of the data used in learning the dictionary, which used to represent the main component of the image. The results show that our technique is as the state-of-the-art in a number of imaging inverse problems, at a lower computational cost. The paper is organized as follows. In sections 2 and 3, we briefly introduce ICA and sparse representation. In section 4, we briefly present modeling of the scenario in decomposing a signal on an overcomplete dictionary in the presence of noise and discuss our algorithm in the real image denoising task. In section 5, we discuss the results of using our algorithm in image denoising. At the end we conclude and give a general overview to future's work.

2. INDEPENDENT COMPONENT ANALYSIS

Independent Component Analysis (ICA) is a higher order statistical tool for the analysis of multidimensional data with inherent data addictiveness property. The noise is considered as Gaussian random variable and the image data is considered as non-Gaussian random variable. Specifically the Natural images are considered for research as they provide the basic knowledge for understanding and modeling of human vision system and development of computer vision systems.

In Gaussian noise, each pixel in the image will be changed from its original value by a (usually) small amount. A histogram, a plot of the amount of distortion of a pixel value against the frequency with which it occurs, shows an estimation of the distribution of noise. While other distributions are possible, the Gaussian (normal) distribution is usually a good model, due to the central limit theorem that says that the sum of independent noises tends to approach a Gaussian distribution. The case of Additive White Gaussian Noise (AWGN) will be considered. The acquired image is expressed in this case in the following form:

$$x = s + n \tag{1}$$

where x is the observed/acquired image, s is the noiseless input image and n is the AWGN component.

Estimating x requires some prior information on the image, or equivalently image models. Finding good image models is therefore at the heart of image estimation.

Some ICA algorithm such as FastICA [6] can be extended to overcomplete problems [22].

In information-theoretic ICA methods [23,24] statistical properties (distributions) of the sources are not precisely known. The learning equation $W \cong A^{-1}(y = Wx)$ has the form:

$$W(\mathrm{k}+1) = W(\mathrm{k}) + \eta[I - E\{\varphi(x)x^T\}]\,W(\mathrm{k}) \tag{2}$$

where is the score function by obtain from:

$$\varphi_i = (\frac{-1}{p_i})(\frac{dp_i}{dx_i}) \tag{3}$$

The unknown density functions p_i can be parameterized, as Generalized K Density (GKD), which is characterized by the following probability density function [25]

$$p(x \mid \alpha, \beta, k) = \frac{\alpha\beta x^{\alpha-1}\exp_k(-\beta x^{\alpha})}{\sqrt{1+k^2\beta^2 x^2\alpha}} \tag{4}$$

where the generalized exponential function $\exp_k(x)$ given by

$$\exp_k(x) = (\sqrt{1+k^2x^2} + kx)^{\frac{1}{k}} \tag{5}$$

where $\alpha > 0$ is a shape parameter, $\beta > 0$ is a scale and $k \in [0,1)$ measures the heaviness of the right tail.

The ICA algorithm in the framework of fast converge Newton type algorithm, is derived using the parameterized generalized k distribution density model. The nonlinear activation function in ICA algorithm is self-adaptive and is controlled by the shape parameter of generalized k distribution density model. To estimate the parameters of such activation function we use an efficient method based on maximum likelihood (ML). If generalized k probability density function is inserted in the optimal form for score function the expression for flexible nonlinearity is obtained:

$$\phi_i = -\frac{(\alpha-1)}{x} + \frac{1}{k}\left(\frac{\frac{\alpha k^2 \beta^2 x^{2\alpha-1}}{\sqrt{1+k^2\beta^2 x^{2\alpha}}} - \alpha\beta k x^{\alpha-1}}{\sqrt{1+k^2\beta^2x^{2\alpha}} - \beta k x^{\alpha}} \right) + \frac{\alpha k^2\beta^2 x^{2\alpha-1}}{1+k^2\beta^2 x^{2\alpha}} \tag{6}$$

The maximum likelihood estimators (MLEs) [26,27] is

$$L(x/\alpha,\beta,k) = \log\prod_{i=1}^{N} p_{x_i}(x_i/\alpha,\beta,k) = (\alpha\beta)^N \prod_{i=1}^{N} \frac{x_i^{\alpha-1}\exp(-\beta x_i^{\alpha})}{\sqrt{1+k^2\beta^2 x_i^2\alpha}} \tag{7}$$

Normally, ML parameter estimates are obtained by first differentiating the log-likelihood function in equation(7) with respect to the generalized k-distribution parameters and then by equating those derivatives to zero (e.g. see [28]). Instead, here we choose to maximize the ML equation in equation (7) by resorting to the Nelder-Mead (NM) direct search method [27] . The appeal of the NM optimization technique lies in the fact that it can minimize the negative of the log-likelihood objective function given in equation (7) essentially without relying on any derivative information. Despite the danger of unreliable performance (especially in high dimensions), numerical experiments have shown that the NM method can converge to an acceptably accurate solution with substantially fewer function evaluations than multi-directional search or steeps descent methods [27]. Good numerical performance and a significant improvement in computational complexity for our estimation method are also insured by obtaining initial estimates from the method of moments. Therefore, optimization with the NM technique to produce the refined ML shape estimates $\hat{\alpha}$ and $\hat{k}$ can be deemed as computationally efficient. Also, an estimate for parameter $\hat{\beta}$ can be calculated for known $\hat{\alpha}$ and $\hat{k}$

$$\hat{\beta} = \frac{1}{2k}\left[\frac{\Gamma(\frac{1}{\alpha})\Gamma(\frac{1}{2k} - \frac{1}{2a})}{k + \alpha\Gamma(\frac{1}{2k} + \frac{1}{2a})} \right]^{\alpha} \tag{8}$$

3. SPARSE REPRESENTATION AND DICTIONARY LEARNING

Sparse representations for signals become one of the hot topics in signal and image processing in recent years. It can represent a given signal $x \in R^n$ as a linear combination of few atoms in an overcomplete dictionary matrix $A \in \mathbb{R}^{n\times k}$ that contains k atoms $\{a_i\}_{i=1}^{k}$ (k>n). The representation of x may be exact $x = As$ or approximate, $x \approx As$,satisfying $\| x - As \|_p \le \varepsilon$, where the vector s is the sparse representation for the vector x .To find s we need to solve either

$$(P_0)\ \min_s \|s\|_0 \text{ subject to } x = As \tag{9}$$

Or

$$(P_{0,\varepsilon})\ \min_s \|s\|_0 \text{ subject to } \|x - As\|_2 \le \varepsilon \tag{10}$$

where $\| \|_0$ is the ℓ_0 norm, the number on non-zero elements.

In this paper we use an ICA based algorithm to learn the basis of an overcomplete dictionary. Like the known K-SVD algorithm but instead of using the SVD decomposition for dictionary atoms update we used the FastICA algorithm with nonlinearity from the Generalized K Distribution for sparse representation for the data matrix. Also we choose the Gabor dictionary as an initial dictionary.

4. ICA FOR OVERCOMPLETE DICTIONARY LEARNING

ICA can be efficient in dictionary learning. Because ICA is most often applied for solving instantaneous Blind Source Separation (BSS) problem:

$$x = As \ , \ A \in R^{N \times M} , \mathrm{s} \in R^{N \times T} \tag{11}$$

Classical ICA methods solve complete (determined and over-determined) BSS problems: $M \leq N$. That was one of the main arguments against using ICA for dictionary learning. Overcomplete dictionary is of practical interest because results in denoising can be better when dictionary is overcomplete (a frame).

In comparison with the probabilistic framework to basis learning in [29], that in partis also based on the use of ICA, the use of ICA proposed here is motivated by two reasons:

1. It extends the probabilistic framework to learn the overcomplete basis, this is achieved through the use of the FastICA algorithm, [12], that works in sequential mode.

2. In regard to the probabilistic framework to basis learning presented in [29], the adopted ICA approach is more flexible, this is due to the fact that proper selection of the nonlinear functions (that are related to parameterized form of the probability density functions of the representation) enables basis learning that is tied with a representation with the pre-specified level of sparseness without affecting the structure of the basis learning equation (by ICA the basis inverse is actually learned).

As opposed to that, in the Bayesian paradigm to the basis learning presented in [29], the structure of the basis learning equation depends on the choice of what was previously imposed on the probability density function of the sparse representation coefficients. We suppose that the linear model $y = D \ x$ is valid; where y and x are random vectors (we interpret columns of the data matrix Y , denoted as y_i , as realizations of y), and D is the basis matrix we want to estimate. For now we consider only the complete case (D is a $n \times n$ square matrix, and y and x are n dimensional).Hence, the basis D is what in blind source separation is referred to as a mixing matrix. Extraction of the code matrix X (also referred to as a source matrix in blind source separation) can be performed by means of the ICA algorithms.

Herein, we are interested in the ICA algorithm that:

1. Can be casted into the probabilistic framework tied with the linear generative model as in[29].

2. Can be extended for learning the overcomplete basis.

When blind source separation problem, $y = D \ x$, the minimization of the mutual information $I(x)$ is used:

$$I(x) = \sum_{i=1}^{n} H(x_i) - H(y) - \log\left|\det D^{-1}\right| \qquad (12)$$

where $H(x_i)$ stands for the differential entropy of the representation and $H(\mathrm{y})$ stands for the joint entropy of the data.

The ICA algorithms that maximize information flow through nonlinear network (Infomax algorithm), maximize likelihood (ML) of the ICA model $y = Dx$, or minimize mutual information between components of $x = D^{-1}y$, are equivalent in a sense that all minimize $I(x)$ and yield the same learning equation for D^{-1}.

$$D^{-1}(i+1) \leftarrow D^{-1}(i) + \eta[I - \phi(x(k)x(i)^T]D^{-1}(i) \qquad (13)$$

If the generalized k probability density function is inserted in the optimal form for score function the expression for flexible nonlinearity is obtained by Equation (6).This enables learning the basis matrix D that gives sparse representation for y_i .For learning an overcomplete dictionary basis we used the FastICA algorithm with the nonlinearity obtained from the GKD. Thus, nonlinear function in the FastICA algorithm can be also chosen to generate sparse distribution of the representation x_i . In the experiments we have used the nonlinearity comes from the GKD, which models sparse or super-Gaussian distributions.

In the sequential mode of the FastICA, basis vectors are estimated one at a time. After every iteration, the basis vector is orthogonalized with respect to previously estimated basis vectors using the Gram-Schmidt orthogonalization. This idea can be extended to over complete case as follows:

$$d_i \leftarrow d_i - \alpha \sum_{j=1}^{i-1} (d_i^T d_j) d_j \qquad (14)$$

and the dictionary updated using equation (13), where ϕ_i represents the score function defined as equation(6).

Reconstruction: reconstruct the denoised image $\hat{x} = D^{-1}y$.

5. EXPERIMENTS AND RESULTS

In this work, the underlying dictionary was trained with the new ICA technique, we used an overcomplete Gabor dictionary as an initial dictionary of size 64×256 generated by using Gabor filter basis of size 8×8, each basis was arranged as an atom in the dictionary. The dictionary then learned and updated by using the proposed algorithm in section 4. We applied the algorithm to images, mainly of size 256×256 and 512×512 with different noise levels, "Lena" and “Barbra” images. The results showed that using the overcomplete dictionary learned by using the FastICA gave a good results. To evaluate our method we calculate the PSNR for denoised BARBRA and LENA images using our method, K-SVD method, and Clustered-based Sparse Representation (CSR) [30]. The comparison results between the three methods are shown in figure 1 and figure 2. The results of the overall algorithm for the images ”Barbara” and “Lena” for $\|n\|_2$ = 20 is shown in Table 1, as it is seen, when the level of noise grows, our approach outperforms K-SVD with OMP and CSR methods. We can conclude that the mentioned algorithms are suitably designed for noisy cased with known low energy.

FIGURE 1: From left to right: original image, noisy image with zero-mean white Gaussian noise of $\|n\|_2$ = 20, the cleaned image via ICA based sparse representation described.

FIGURE 2: From left to right: original image, noisy image with zero-mean white Gaussian noise of $\|n\|_2 = 20$, the cleaned image via ICA based sparse representation described.

Sigma	BARBARA			LENA		
	K-SVD	ICA based	CSR	K-SVD	ICA based	CSR
5	38.08	37.41	37.52	38.60	38.18	38.56
10	34.42	34.51	34.35	35.52	35.42	35.38
15	32.36	32.79	32.45	33.69	33.88	33.62
20	30.83	32.02	30.94	32.38	33.46	32.56
25	29.62	31.05	30.02	31.32	32.72	31.47

TABLE 1: The PSNR computed for Barbra image and Lena image with different noise variance level (sigma).

6. DISCUSSION AND CONCLUSION

ICA-learned dictionary yields good or favorable results when compared against other methods. Yet, the ICA-based dictionary learning is faster than those by competing methods. It appears that ICA-learned dictionary is less coherent than the dictionary learned by K-SVD and the sparsity based structural clustering (CSR) on the same training set.

In this paper a simple algorithm for denoising application of an image was presented leading to state-of-the-art performance, equivalent to and sometimes outperform recently published leading alternative. We addressed the image denoising problem based on sparse coding over an overcomplete dictionary. Based on the fact that the ICA can capture the most important component of real data, which implies on real images. We presented our algorithm, which used the technique of learning the dictionary to be suitable for representing the important component in the image by using the FastICA technique that uses the nonlinearity induced from the Generalized K Distribution (GKD) for updating the dictionary in the learning process. Experimental results show satisfactory recovering of the source image. Moreover, for our technique, the larger the noise level is, the better the effect on the denoising results is.

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Design of Low-Pass Digital Differentiators Based on B-splines

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Abstract

This paper describes a new method for designing low-pass differentiators that could be widely suitable for low-frequency signals with different sampling rates. The method is based on the differential property of convolution and the derivatives of B-spline bias functions. The first order differentiator is just constructed by the first derivative of the B-spline of degree 5 or 4. A high (>2) order low-pass differentiator is constructed by cascading two low order differentiators, of which the coefficients are obtained from the *n*th derivative of a B-spline of degree *n*+2 expanded by factor *a*. In this paper, the properties of the proposed differentiators were presented. In addition, we gave the examples of designing the first to sixth order differentiators, and several simulations, including the effects of the factor *a* on the results and the anti-noise capability of the proposed differentiators. These properties analysis and simulations indicate that the proposed differentiator can be applied to a wide range of low-frequency signals, and the trade-off between noise-reduction and signal preservation can be made by selecting the maximum allowable value of *a*.

Keywords: Low-pass Differentiator, B-spline, Finite-impulse Response (FIR), Digital Filters.

1. INTRODUCTION

Digital differentiators (DDs) have been applied in several areas, such as radar, sonar, communication systems and signal processing system [1-3]. In particularly, low-pass high-order DDs are utilized in biological and electrochemical signal processing etc [4, 5]. Since the signal values are known on discrete points because of sampling operation, difference approximation is usually used to design DDs [6]. However, differentiation could amplify the noises contaminating the signal, especially the high-frequency noises [4, 5, 7]. And the signals we need to study, such

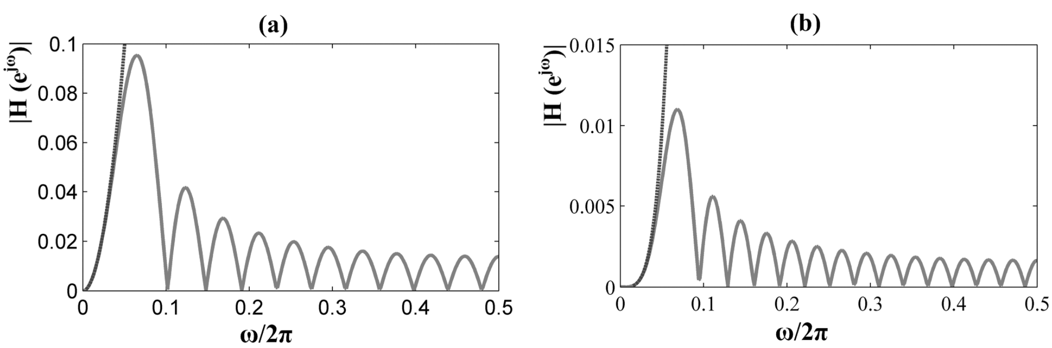

FIGURE 1: The frequency responses of the Savitzky-Golay digital differentiators (SGDDs) (—) and the corresponding ideal differentiators (---). (a) is the frequency response of the 2nd order SGDD by using fitting coefficients of fourth-order polynomials on 25 points, and (b) is the frequency response of the 4th order SGDD by using fitting coefficients of sixth-order polynomials on 35 points.

as biological and electrochemical signals, are mostly at low frequencies. Therefore, low-pass digital differentiators (LPDDs) have been to estimate the derivatives [5, 7, 8].

Many methods have been available for the design of LPDDs. Most of them focus on the first order differentiators [8-10], which cannot directly obtain the high order derivatives of the signals. The Savitzky-Golay digital differentiators (SGDDs) are generally used for smoothing and acquiring low or high order derivatives due to their low-pass characteristic and arbitrary lengths etc [5]. But there are several weak points for SGDDs. One hand, the filter length and the degree of fitting polynomials can be arbitrarily selected, which instead, makes it blind in selections. Although, recent researches have focused on adaptive extension of the SG approach [11-13], they may increase the complexity of the algorithm, and there is still a need for further researches and tests. On the other hand, the frequency responses of SGDDs have several ripples at high frequencies, and the frequency responses of even order SGDDs at $\omega = \pi$ are not zero (Figure 1), which may affect the results of SGDDs filtering the high-frequency noises [5, 14].

In order to meet the low-pass characteristic and apply to different types of signals, we propose a method for designing LPDDs based on B-splines by using the differential property of convolution. B-splines have been widely used in data smooth because of their explicit formulae and Gaussian-like waveforms [15, 16]. Moreover, the derivatives of B-spline bias functions are continuous and easily obtained. Consequently, B-splines have been used to calculate the derivatives of the gray of the image [15, 17]. However, they have not been widely used to obtain the derivatives of sampled signals.

The aim of this study, therefore, was to propose a method for designing any order LPDDs, which were simple, flexible, easy to control and suitable for low-frequency signals with various sampling rates. In this paper, we first introduced the method of the designs of LPDDs. Then, several properties of the proposed LPDDs were summarized, and some computer simulations of various orders LPDDs, acting on the input testing signals produced by a Gaussian function in different ways, were also presented.

2. THEORIES

2.1 Background

Let β_m denote the mth order central B-spline function that can be generated by repeated convolutions of a B-spline of degree 1

$$\beta_{\mathrm{m}}(t) = \beta_{\mathrm{m\text{-}1}}(t) * \beta_1(t) \tag{1}$$

where $\beta_1(t)$ is the indicator function in the interval [-1/2, 1/2], and the derivatives of central B-splines can be obtained in a recursive fashion based on the following property [15, 18]:

$$\frac{d\beta_{\mathrm{m}}(t)}{d\mathrm{t}} = \beta_{\mathrm{m\text{-}1}}(t+\frac{1}{2}) - \beta_{m-1}(t-\frac{1}{2}) \tag{2}$$

If $f(t)$ denotes a continuous signal, and $\beta_m(t/a)$ is the B-spline of degree m expended by factor a, the convolution between $f(t)$ and the nth derivative of $\beta_m(t/a)$ could be written as:

$$\begin{aligned} W(t) &= f(t) * \beta_{\mathrm{m}}^{(n)}(t/a) = f(t) * \frac{d^n}{d(t/a)^n}\beta_m(t/a) \\ &= f(t) * \left[a^n \cdot \frac{d^n}{dt^n}\beta_m(t/a)\right] \\ &= a^n \cdot \int \beta_m(t/a)\mathrm{d}t \cdot \frac{d^n}{dt^n}\left[\frac{f(t) * \beta_m(t/a)}{\int \beta_m(t/a)\mathrm{d}t}\right] \\ &= a^{n+1} \cdot \frac{d^n}{dt^n}\left[f(t) * \beta_m(t/a)/a\right] \end{aligned} \tag{3}$$

which is based on the differential property of convolution. The B-spline functions become more and more Gaussian-like with the degree m increasing [15], and therefore the convolution between $f(t)$ and $\beta_m(t/a)$ is really to smooth $f(t)$ by $\beta_m(t/a)$ in (3). Accordingly, $W(t)$ could be taken as the dilation of the nth derivative of $f(t)$ smoothed by $\beta_m(t/a)$.

When the signal $f(t)$ is sampled once every T seconds, (3) could be written as:

$$\begin{aligned} W(t) &= \mathrm{f}(t) * \beta_{\mathrm{m}}^{(n)}(t/a) = \int f(\tau) * \beta_m^{(n)}(\frac{t-\tau}{a})d\tau \\ &\approx T \times \sum_i f(iT) \cdot \beta_m^{(n)}[(j-i)T/a] \end{aligned} \tag{4}$$

where T is the sampling interval, i represents the sample number, and $jT = t$. Since $\beta_m(kT/a)$ is the discrete representation of $\beta_m(t/a)$, the discrete Fourier transform of the sequence $\{\beta_m(kT/a)\}$ and the corresponding nth derivative are respectively defined as $\Phi_m(e^{j\omega})$ and $U_{m,n}(e^{j\omega})$. Figure 2 shows the frequency responses of $\Phi_m(e^{j\omega})$ depending on the values of a and m. When a and m increase, the 3 dB cut-off frequency decreases, and the effect of low-pass filtering tends to be more noticeable. In Appendix A, we prove that the 3 dB cut-off frequency f_c of $\Phi_m(e^{j\omega})$ is independent of sampling frequency when a and m are constants. Moreover, for a given B-spline of degree m, the relationship between f_c and a displays as following (see Appendix B):

$$f_c \cdot a = f_m \tag{5}$$

where the value of f_m is only determined by the degree of the B-spline bias function. According to (3), we also know that the 3 dB cut-off frequency of differentiators designed by $\beta^{(n)}{}_m(t)$ is actually the 3 dB cut-off frequency f_c of $\Phi_m(e^{j\omega})$.

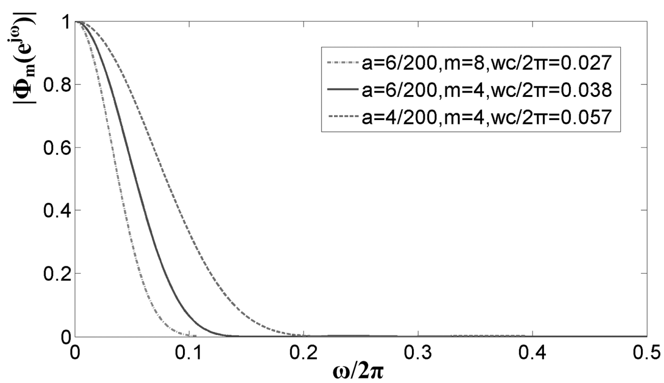

FIGURE 2: The frequency responses of $\Phi_m(e^{j\omega})$ for different values of factor a and degree m when the sampling frequency is 200 Hz.

2.2 The Designs of LPDDs

(3) tells us how to obtain the nth derivative of a signal. Obviously, a B-spline of high degree can well smooth the signal $f(t)$ by convolution. However, it may filter some useful information contained in the signal, and need larger computations at the same time. To make the differentiators easy and avoid complicated computations, we design the LPDDs by cascading two low order differentiators.

Usually, the nth derivative of a B-spline of degree n+2 comprises of piecewise linear polynomials (see Appendix C), which could construct each of the two low order differentiators. Table 1 shows the designs of the 2nd to 6th order differentiators by cascading two low order differentiators, and the first order differentiator is just constructed by the first derivative of the B-spline of degree 5 or 4. In the progress of the algorithm implementation, $f(t)$ is convoluted with the coefficients of the two low order differentiators by using the associative property of convolution as shown in (6).

Although it seems like that the two consecutive convolution operations increase the computational cost, the convolution operations between $f(t)$ and the two low order differentiators avoids calculating the polynomial of high degree in t because the low order differentiator is constructed by piecewise linear polynomials. Moreover, different combinations of several low order differentiators could construct more high order LPDDs.

$$\mathrm{f}(t) * \left[\beta_{m1}^{(n1)}(\frac{t}{a}) * \beta_{m2}^{(n2)}(\frac{t}{a})\right] = \left[f(t) * \beta_{m1}^{(n1)}(\frac{t}{a})\right] * \beta_{m2}^{(n2)}(\frac{t}{a}) \tag{6}$$

If $\{f(iT)\}$ is the input sequence of a single differentiator of degree n, and $\{y[j]\}$ is normalized output, the relationship between $y[j]$ and $f(iT)$ displays in (7).

$$y[j] = (T/a)^{n+1} \cdot \sum_{i=-N}^{N} \beta_m^{(n)}(\frac{iT}{a}) f[(j-i)T] \tag{7}$$

$$\beta_{m1}^{(n1)}(t/a) * \beta_{m2}^{(n2)}(t/a) = a \cdot \beta_{m1+m2}^{(n1+n2)}(t/a) \tag{8}$$

where the value of a is usually a multiple of T, and N is the largest integer less than $a{\cdot}m/(2T)$. In addition, (8) is derived by the convolution property. So the 3 dB cut-off frequency of the two cascading low order differentiators is just that of the differentiator constructed by $\beta^{(n1+n2)}_{(m1+m2)}(t)$, and is also equal to that of the filter constructed by the $\beta_{m1+m2}(t)$. By using (5), we get the corresponding equations of a and f_c as shown in Table 1.

The Order of Differentiators	Designs of Differentiators	The Relationship Between a and f_c
1	$\beta_4^{(1)}(t/a)$	$f_c \times a = 0.228$
1	$\beta_5^{(1)}(t/a)$	$f_c \times a = 0.204$
2	$\beta_3^{(1)}(t/a) * \beta_3^{(1)}(t/a)$	$f_c \times a = 0.187$
3	$\beta_3^{(1)}(t/a) * \beta_4^{(2)}(t/a)$	$f_c \times a = 0.173$
4	$\beta_4^{(2)}(t/a) * \beta_4^{(2)}(t/a)$	$f_c \times a = 0.162$
5	$\beta_4^{(2)}(t/a) * \beta_5^{(3)}(t/a)$	$f_c \times a = 0.152$
6	$\beta_5^{(3)}(t/a) * \beta_5^{(3)}(t/a)$	$f_c \times a = 0.145$

TABLE 1: The designs of low-pass digital differentiators (LPDDs) and the relationship between the factor a and the 3 dB cut-off frequency f_c of the corresponding B-spline filters.

3. THE PROPOSED DIFFERENTIATORS

3.1 Low-pass Characteristic

Using (7) and (8), we obtain the frequency response of the differentiator of degree n ($n = n1+n2$).

$$H_1(\mathrm{e}^{j\omega}) = T^2 \cdot a^{-2} \cdot \sum_{i=-\infty}^{+\infty} \beta_m^{(1)}(iT/a) e^{-j\omega i} \quad n = 1 \tag{9}$$

$$\begin{aligned} H_n(\mathrm{e}^{j\omega}) &= H_{n1}(\mathrm{e}^{j\omega}) \cdot H_{n2}(\mathrm{e}^{j\omega}) \\ &= \left[T^{n1+1} \cdot a^{-n1-1} \cdot \sum_{i=-\infty}^{+\infty} \beta_{m1}^{(n1)}(iT/a) e^{-j\omega i}\right] \cdot \quad n > 1 \\ &\left[T^{n2+1} \cdot a^{-n2-1} \cdot \sum_{i=-\infty}^{+\infty} \beta_{m2}^{(n2)}(iT/a) e^{-j\omega i}\right] \end{aligned} \tag{10}$$

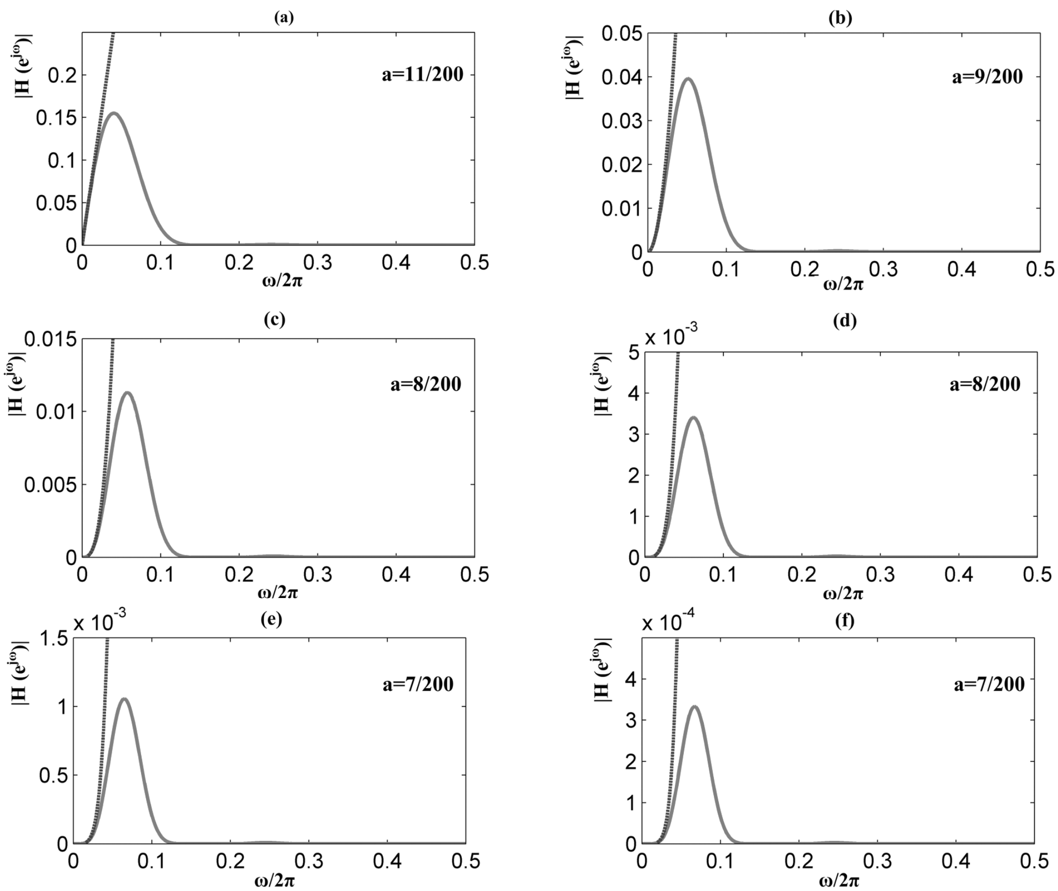

FIGURE 3: The solid lines are the frequency responses of the proposed differentiators of degree 1 (a), degree 2 (b), degree 3 (c), degree 4 (d), degree 5 (e), and degree 6 (f) when the sampling frequency is 200 Hz. The dotted lines are the those of the corresponding ideal differentiators.

As can be seen from Figure 3, the amplitude of $H_n(e^{j\omega})$ is close to the frequency response of the ideal differentiator at low frequencies, and rapidly decays to zero with few ripples, making the differentiator filter high-frequency noises effectively.

3.2 Flexible and Easy to Control

The cut-off frequency is one of the key parameters of a filter. (5) and (8) indicate that the cut-off frequency of a proposed *n*th order differentiator is only determined by *a*. Knowing the effective frequency band of the sampled signal, we can use the equation of 3 dB cut-off frequency and *a* shown in Table 1 to obtain the maximum value of *a*, which is usually a multiple of *T*.

3.3 Impulse Response Restriction

If $h_n[i]$ denotes the impulse response sequence of a single differentiator of degree *n*, using (9) and (10), we derive

$$h_n[i] = (T / a)^{n+1} \cdot \beta_m^{(n1)}[(i - N)T / a] \quad i = 0, 1 \ldots 2N \quad (11)$$

Obviously $\{h_n[i]\}$ is a finite-length sequence. If a *n*th order differentiator constructed by cascading two low order differentiators, of which degree are respectively *n*1 and *n*2, we can get the impulse response sequence

$$h_{n1+n2}[i] = h_{n1}[i] * h_{n2}[i] \qquad (12)$$

Similarly, $\{h_{n1+n2}[i]\}$ is also a finite-length sequence. Therefore, the proposed differentiators based on B-splines are finite impulse response differentiators. This property is consistent with B-spline filters [15].

3.4 A Low-Complexity Algorithm

According to (6), calculating the derivative of $f(t)$ is just the discrete convolution between $f(iT)$ and $\beta^{(n1)}_{m1}(iT/a)$ without any other filtering algorithms. Moreover, the process of convolutions avoids calculating the polynomial of high degree in t, for $\beta^{(n1)}_{m1}(iT/a)$ comprises of piecewise linear polynomials.

3.5 A Flexible And Easy To Control Frequency Response Flatness At ω = 0

The frequency response of the ideal full-band nth order DD is [9, 14]

$$H_{FBn}(e^{j\omega}) = (j\omega)^n \tag{13}$$

As illustrated in Appendix D, the frequency response of the proposed differentiator of degree n satisfies the flatness constraints

$$\left|H_n(e^{j\omega})\right| = 0 \quad \omega = 0 \tag{14}$$

$$\frac{d}{d\omega}\left|H_n(e^{j\omega})\right| = n! \quad \omega = 0 \tag{15}$$

which is consistent with the frequency response of the proposed differentiators close to the ideal DD at low frequencies as shown in Figure 3.

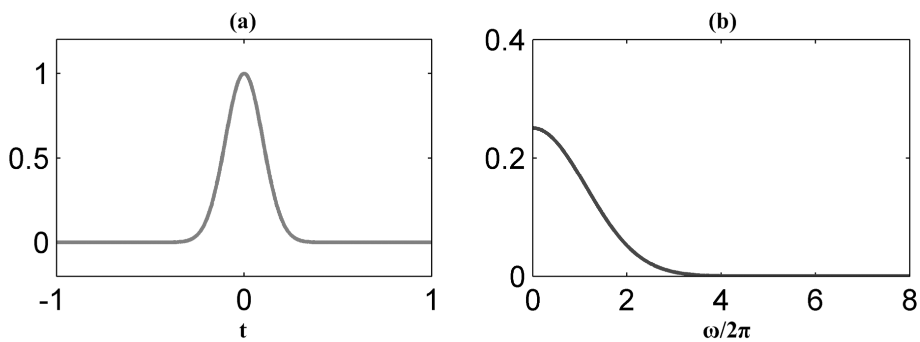

FIGURE 4: The input testing signal (a) and its Fourier transform (b).

4. SIMULATIONS AND EXPERIMENTS

4.1 The Input Testing Signal

Usually, the performance of a differentiator is evaluated by simulating Gaussian signals[5, 14, 19]. In our study, a Gaussian pulse function $g(t) = exp(-50t^2)$ sampled every 5 milliseconds, as depicted in Figure 4, was taken as the input testing signal. According to the Fourier transform of the input testing signal (Figure 4), we found that the frequency of the input signal containing the major components is maintained below 4 Hz.

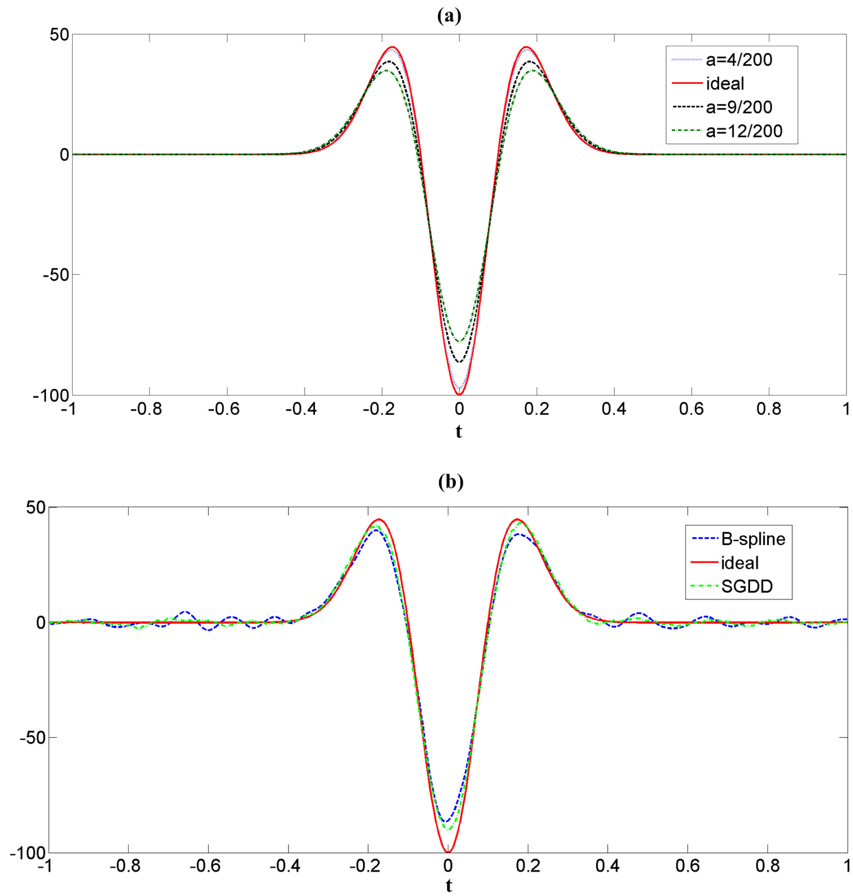

FIGURE 5: (a)The ideal 2nd derivative (—) of the input testing signal, and the 2nd derivatives using the proposed 2nd order differentiator at different values of factor *a*; (b) The ideal 2nd derivative (—), the 2nd derivative of the input testing signal contaminated by Gaussian white noise using the 2nd order Savitzky-Golay digital differentiator (SGDD) (–·), and the 2nd derivative using the 2nd order proposed differentiator (---).

Values of factor *a*	The *t* value of the first peak	the *t* value of the second peak	The first zero-crossing point	The second zero-crossing point
4/200	-35/200	35/200	-20/200	20/200
6/200	-36/200	36/200	-20/200	20/200
7/200	-36/200	36/200	-21/200	21/200
8/200	-36/200	36/200	-21/200	21/200
9/200	-36/200	36/200	-21/200	21/200
10/200	-37/200	37/200	-21/200	21/200
12/200	-38/200	38/200	-22/200	22/200

TABLE 2: The positions of characteristic points of the 2nd derivative waveform of the testing signal obtained by the proposed 2nd order differentiator.
Note: the corresponding positions of characteristic points of the ideal 2nd derivative waveform of the input testing signal respectively are -35/200, 35/200, -20/200, 20/200.

4.2 The Optimal Factor a

The method of choosing optimal factor *a* was displayed by giving an example. Using table 1, we know that the maximum value of factor *a* is 9/200 when the 3 dB cut-off frequency of the proposed 2nd order differentiator is not less than 4 Hz. Figure 5a displayed the ideal second derivative of the input testing signal and the waveforms derived by the proposed second order differentiator at different values of factor *a*. Additionally, according to Table 2, we found that it could get good signal-preservation when *a* is not more than 9/200.

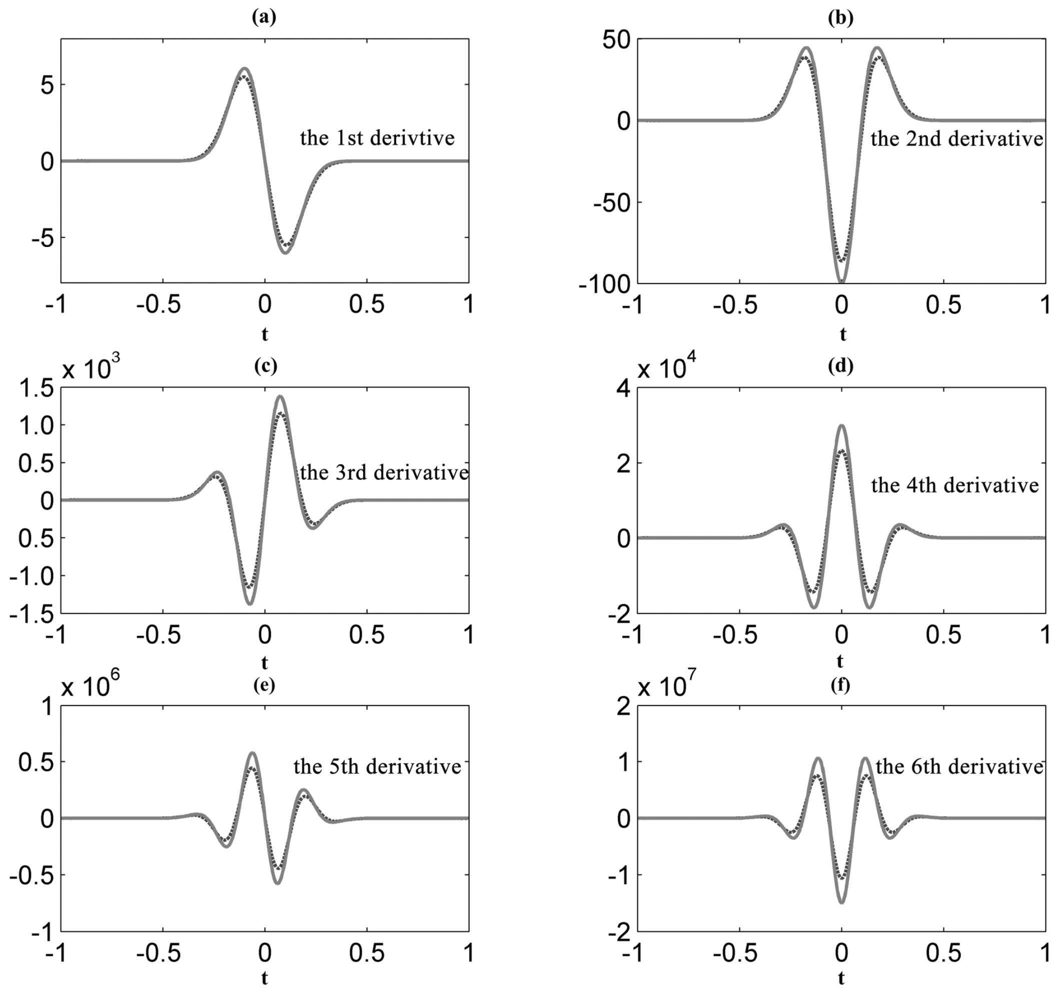

FIGURE 6: The first to sixth order derivatives of the input testing signal using the proposed differentiators (---), and corresponding ideal derivatives (—).

4.3 The Anti-Noise Capability Of The Proposed Differentiators

The anti-noise capability was evaluated by adding uncorrelated Gaussian white noise with signal-to-noise ratio (SNR) = 28.5 dB to the input testing signal. The results derived by the proposed second order differentiator at factor a = 9/200 and the second order SGDD by using fitting coefficients of fourth-order polynomials on 69 points were compared. The 3 dB cut-off frequencies of the two differentiators were both about 4.2 Hz. As can be seen from Figure 5b, both the proposed differentiator and the SGDD could restrain the high frequency noises, and the proposed differentiator got a smoother waveform than the SGDD.

4.4 The First To Sixth Derivatives Of The Input Signal

Several derivative waveforms of the input testing signal are used to validate the feasibility of the proposed differentiators. Figure 6 displayed the first to sixth derivatives of the input testing signal obtained by the proposed differentiators at the maximum value of factor *a*.

5. DISCUSSION

This study has presented a new method for designing LPDDs based on B-splines, where some examples have been used to validate the reliability and the anti-noise capability of the proposed differentiators. The proposed differentiators have some good properties, making them have some advantages in obtaining the derivatives of input signals. One is that the value of *a* could be adaptively selected by calculating the maximum allowable value of *a*. Another is that the trade-off between noise-reduction and signal preservation can be made by selecting the maximum allowable value of *a*, when the 3 dB cut-off frequency of differentiators is equal to that of the sampled signal. In addition, the cut-off frequency of the proposed differentiators is independent of sampling rate. Therefore, the proposed differentiators could be applied to a wide range of low-frequency signals.

The value of factor *a* could be adaptively selected. According to the Fourier transform of the Gaussian function, the input testing signal components cover the entire frequency range [20]. The higher the frequency is, the fewer components of the signal are distributed. Most of the signal components are maintained below 4 Hz. Therefore, when the value of *a* is low, the proposed differentiator could preserve more information of the input testing signal because of its high 3 dB cut-off frequency (as shown in Figure 2). As *a* reaches the maximum allowable value, the 3 dB cut-off frequency of differentiators is equal to or close to the maximum significant frequency of the input signal. In this case, the differentiator could filter out more of the high frequency components of the input signal, which displays by the differences of amplitudes of peaks and troughs in the two waveforms (Figure 5a). However, the differences of the positions of peaks, troughs, and the zero-crossing are little (Table 2). This illustrates the proposed differentiators could preserve signal's original features when *a* is not more than the maximum allowable value (Figure 6).

Our differentiators could easily get the trade-off between noise-reduction and signal preservation. Numerous approaches of differentiators designs have previously been displayed. the SGDD is currently one of the most common differentiators, and also is a finite impulse response LPDD [5, 12]. The SGDDs have many excellent properties [5, 21], but their frequency response have several ripples at high frequencies (Figure 1), which may affect the results of SGDDs filtering the high-frequency noises. By contrast, the frequency response of the proposed differentiators have few ripples (Figure 3), reducing almost all of the high-frequency noises. When *a* reaches the maximum allowable value, the proposed differentiator could get trade-off between noise-reduction and signal preservation. That explains the waveform derived by the proposed 2nd order differentiators is smoother than that of the 2nd order SGDD (as shown in Figure 5b). In addition, the only one key parameter of the proposed differentiator is the factor *a*, which directly determines the 3 dB cut-off frequency of differentiators. Then, using the equation between factor *a* and the 3 dB cut-off frequency, we could get the value of *a* according to the characteristics of the input signal. This makes the proposed differentiators flexible and easy to control, avoiding the work of selecting parameters of the SGDDs by testing [13].

This study proposed a easy method for designing high order LPDDs. There have been several studies on the designs of the first order LPDDs [23, 24]. However, high order LPDDs can only be constructed by cascading the first order LPDDs one by one in these studies. Therefore, the designed high order differentiators are very complicate. The proposed high order LPDD can be constructed by cascading only two low order LPDDs, and the coefficients of the differentiator can be easily acquired.

Finally, it is important to note that the cut-off frequency of the proposed differentiator is independent of sampling rate according to (5). Consequently, our differentiators are not limited by the signals with a wide range of applications. In addition, further studies in reducing the transition band of the proposed differentiator are required to improve the anti-noise capability of the differentiator.

6. CONCLUSION

The primary goal of this paper is to introduce a method for designing any order LPDDs. Several examples of the designs of the first to sixth order differentiator and some simulations were presented to validate the feasibility of this method. All these properties analysis and simulations indicate that the differentiators designed by the proposed method could be well suitable for different types of low-frequency signals, and the trade-off between noise-reduction and signal preservation could be made by selecting the maximum allowable value of *a*. But yet the behavior of the proposed differentiators needs to be tested in a wide range of situations, including the applications in reality. Further work also needs to concentrate on reducing the transition bandwidth to further improve the anti-noise capability of the proposed differentiators, especially for high order differentiators.

7. APPENDIX

7.1 Appendix A

Let $g[n]$ denote the discrete representation of $\beta_m(t)$

$$g[n] = \beta_m(n \cdot T) \qquad \text{(A.1)}$$

where T designates the sampling period. The Fourier transform of $\beta_m(t)$ and the discrete transform of $g[n]$ are

$$S(\omega) = \int_{-\infty}^{+\infty} \beta_m(t) e^{-j\omega t} dt \approx \sum_{n=-\infty}^{+\infty} \beta_m(nT) e^{-j\omega nT} / T \qquad \text{(A.2)}$$

$$\begin{aligned} G(\mathrm{e}^{j\omega}) &= \sum_{-\infty}^{+\infty} g[n] e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} \beta_m(nT) e^{-j\omega n} \\ &= S(\omega / T) / T \end{aligned} \qquad \text{(A.3)}$$

Now let f_m, ω_c respectively denote the 3 dB cut-off frequency of $S(\omega)$ and the normalized cut-off angular frequency of $G(e^{j\omega})$.

$$|S(2\pi f_m)| = |S(0)| / \sqrt{2} \qquad \text{(A.4)}$$

$$\left| G(e^{j\omega_c}) \right| = |S(\omega_c / T)| / T = S(0) / (\sqrt{2} T) \qquad \text{(A.5)}$$

Using (A.4) and (A.5), we can derive

$$\omega_c = 2\pi f_m \cdot T \qquad \text{(A.6)}$$

of which the corresponding ordinary frequency is

$$f_{3dB} = \frac{\omega_c}{2\pi} \cdot f_s = \frac{2\pi f_m \cdot T}{2\pi} \cdot \frac{1}{T} = f_m \qquad \text{(A.7)}$$

7.2 Appendix B

The Fourier transform of $\beta_m(t/a)$ is given by

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{+\infty} \beta_m(\frac{t}{a}) e^{-j\omega t} dt \\ &= a \cdot \int_{-\infty}^{+\infty} \beta_m(t) e^{-j\omega a t} dt = a \cdot S(a\omega) \end{aligned} \qquad \text{(B.1)}$$

Let f_c denote the 3 dB cut-off frequency of $F(\omega)$. Using (A.4), we derive

$$f_c \cdot a = f_m \qquad \text{(B.2)}$$

7.3 Appendix C

Parameters	Representations
$n = 1$	$\beta_3^{(1)}(t) = \begin{cases} t+1.5, -1.5 \le t \le -0.5 \\ -2t, -0.5 < t \le 0.5 \\ -t-1.5, 0.5 < t \le 1.5 \end{cases}$
$n = 2$	$\beta_4^{(2)}(t) = \begin{cases} 3\lvert t\rvert - 2, \lvert t\rvert \le 1 \\ -\lvert t\rvert + 2, 1 < \lvert t\rvert \le 2 \end{cases}$
$n = 3$	$\beta_5^{(3)}(t) = \begin{cases} t+2.5, -2.5 \le t \le -1.5 \\ -4t-5, -1.5 < t \le -0.5 \\ 6t, -0.5 < t \le 0.5 \\ -4t+5, 0.5 < t \le 1.5 \\ -t-2.5, 1.5 < t \le 2.5 \end{cases}$
$n = 4$	$\beta_6^{(2)}(t) = \begin{cases} -10\lvert t\rvert + 6, \lvert t\rvert \le 1 \\ 5\lvert t\rvert - 9, 1 < \lvert t\rvert \le 2 \\ -\lvert t\rvert + 3, 2 < \lvert t\rvert \le 3 \end{cases}$

TABLE 3: The representations of the *n*th derivative of b-spline bias functions of degree *n*+2.

7.4 Appendix D

Let $H_n(e^{j\omega})$ denote the frequency response of the proposed differentiator of degree *n* that can be written as

$$\begin{aligned} H_1(\mathrm{e}^{j\omega}) &= T^2 \cdot a^{-2} \cdot \sum_{i=-\infty}^{+\infty} \beta_m^{(1)}(iT/a) e^{-j\omega i} \\ &\approx T^2 \cdot a^{-2} \cdot S_1(a\omega/T) \cdot a/T \\ &= (\mathrm{T/a})^1 \cdot S_1(a\omega/T) \end{aligned} \qquad n = 1 \qquad \text{(D.1)}$$

$$\begin{aligned} H_{n1+n2}(\mathrm{e}^{j\omega}) &= H_{n1}\left(\mathrm{e}^{j\omega}\right) \cdot H_{n2}\left(\mathrm{e}^{j\omega}\right) \\ &= \left[T^{n1+1} \cdot a^{-n1-1} \cdot \sum_{i=-\infty}^{+\infty} \beta_{m1}^{(n1)}(iT/a) e^{-j\omega i}\right] \\ &\cdot \left[T^{n2+1} \cdot a^{-n2-1} \cdot \sum_{i=-\infty}^{+\infty} \beta_{m2}^{(n2)}(iT/a) e^{-j\omega i}\right] \\ &\approx \left[T^{n1+1} \cdot a^{-n1-1} \cdot S_{n1}(a\omega/T) \cdot a/T\right] \\ &\cdot \left[T^{n2+1} \cdot a^{-n2-1} \cdot S_{n2}(a\omega/T) \cdot a/T\right] \\ &= (T/a)^n \cdot S_{n1+n2}(a\omega/T) \end{aligned} \qquad n > 1 \qquad \text{(D.2)}$$

$$S_{n1+n2}(\omega) = \int_{-\infty}^{+\infty} \beta_{m1+m2}^{(n1+n2)}(t) e^{-j\omega t} dt \qquad \text{(D.3)}$$

where *n* is the value of *n*1 and *n*2 ($n > 1$), T is the sampling period. The *n*th derivative of $\beta_m(t)$ is derived by [22]

$$\beta_m^{(n)}(t) = \sum_{i=0}^{n} (-1)^i C_n^i \beta_{m-n}(t - n/2 + i) \qquad \text{(D.4)}$$

Setting $m = m1 + m2$, $\omega = 0$, (D.3) becomes

$$\begin{aligned} S_{n1+n2}(\omega)\big|_{\omega=0} &= \int_{-\infty}^{+\infty} \beta_{m1+m2}^{(n1+n2)}(t)dt \\ &= \int_{m/2}^{-m/2} \sum_{i=0}^{n} (-1)^i C_n^i \beta_{m-n}(t-n/2+i)dt \\ &= \sum_{i=0}^{n} (-1)^i C_n^i \int_{-m/2}^{m/2} \beta_{m-n}(t-n/2+i)dt \\ &= \sum_{i=0}^{n} (-1)^i C_n^i \\ &= 0 \end{aligned} \tag{D.5}$$

And the nth derivative of $S_n(\omega)$ at $\omega = 0$ is

$$\begin{aligned} S_{n1+n2}^{(n1+n2)}(\omega)\big|_{\omega=0} &= \int_{-\infty}^{+\infty} \beta_m^{(n)}(t)(-j\omega)^n dt \\ &= (\text{-} \mathrm{j})^n \int_{m/2}^{-m/2} t^n d\beta_m^{(n-1)}(t) \\ &= (\text{-} \mathrm{j})^n \left[t^n \beta_m^{(n-1)}(t)\Big|_{t=-m/2}^{t=m/2} - n \int_{m/2}^{-m/2} t^{n-1} \beta_m^{(n-1)}(t)dt \right] \\ &= (\text{-} \mathrm{j})^n \cdot (-1)^n \cdot n! \cdot \int_{-m/2}^{m/2} \beta_m(t)dt \\ &= j^n \cdot n! \end{aligned} \tag{D.6}$$

Then we can get the frequency response of the proposed differentiator of degree n at $\omega = 0$, as well as the nth derivative of $H_n(e^{j\omega})$ from (D.1), (D.2), and (D.6).

$$\begin{aligned} H_n(\mathrm{e}^{j\omega})\big|_{\omega=0} &= (T/a)^n \cdot S_{n1+n2}(a\omega/T)\big|_{\omega=0} \\ &= 0 \end{aligned} \tag{D.7}$$

$$\begin{aligned} & H_n^{(n)}(\mathrm{e}^{j\omega})\big|_{\omega=0} \\ &= (T/a)^n \cdot (a/T)^n \cdot S_{n1+n2}^{(n1+n2)}(a\omega/T)\big|_{\omega=0} = j^n \cdot n! \end{aligned} \tag{D.8}$$

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