Bid and Ask Prices Tailored to Traders' Risk Aversion and Gain Propension: a Normative Approach

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Abstract

Risky asset bid and ask prices "tailored" to the risk-aversion and the gain-propension of the traders are set up. They are calculated through the principle of the Extended Gini premium, a standard method used in non-life insurance. Explicit formulae for the most common stochastic distributions of risky returns, are calculated. Sufficient and necessary conditions for successful trading are also discussed.

Keywords: Extended Gini Index, Bid and Ask prices, Pessimism and Optimism indices.

1. INTRODUCTION

An *evergreen* debate in risk management concerns the pricing of risky assets. Clearly, the perception of the asset return variability not only depends on the individual risk-aversion and/gain-propension of the trader but also if she is acting on the market as a buyer or as a seller of the asset. Specifically, we define the bid-price as the highest price at which a buyer will buy a risky position, whereas the ask-price (or short-selling price) as the lowest one at which she will sell it short.

The aim of the paper is to propose a normative approach to calculate personalized bid and ask prices as Extended Gini (EG) premia. The definition of EG is seminally due to Yitzhaki ([1]) as an extension of the Gini index, the most popular measure of income inequality (see [2]). Although its originally use was confined to the social welfare context, recently it has become familiar in non-life insurance to determine the safety loading in insurance premia (see [3]), in Quantitative Finance to calculate risk measures through the Lorenz curve (see Shalit and Yitzhaki [4], [5] and[6]), and in portfolio risk management to select the best tradeoffs between risk and returns (see [7] and [8]). Technically speaking, these prices can be represented as Choquet integrals with power-type distortion functions of order given by the agent index of pessimism and optimism, respectively (see [9] and [10]).

After having set up closed-end formulae for the most common stochastic distributions used in risk management, we discuss sufficient and necessary conditions for successful trading.

To quantify the chance of successful in trading we suggest to use the so-called probability of trading. We show that the willingness to trade is driven by three factors (1) the buyer risk-aversion, (2) the seller gain-propension and (3) the return distribution and specifically, its asymmetry. We get a confirmation of what intuition suggests: (1) the higher the buyer risk-aversion and the seller gain-propension, the higher the probability of no-trading; (2) if the density function of the asset return is decreasing (then the distribution is right-skewed) the probability of buying is higher than that of selling; and vice versa if the density function is increasing; if the asset is symmetrically distributed, the probability of buying and selling are equal. Among the distributions studied, the uniform distribution has the smallest probability of no-trade and the normal distribution has the largest one. However, for some distributions, one can have probability one or arbitrarily close to zero of successful trading.

The remainder of this paper is organized as follows. Section 2 introduces the bid and ask prices definition. In Section 3 closed-formulae and tables for the most common distributions are given. In Section 4 we discuss sufficient and necessary conditions for trading. Then we define the probability of trading and we discuss the impact of the asset skewness on the successful trading. Section 5 concludes the paper. An Appendix collects the proofs.

2. BID AND ASK PRICES AS RISK-ADJUSTED MEAN RETURN

In social welfare studies, a celebrated measure for income inequality is the Gini index that is well suited for long-tailed distributions such as those of incomes. In finance context, Yitzhaki (1982) seminally developed an extension, called the Extended Gini (EG) and subsequently thoroughly studied by Shalit and Yitzhaki S. ([4], [5] and [6]), Shalit [7] and Cardin et al. [8] among others.

The definition of EG in the literature is not unique, but they all coincide in the continuous case. To avoid technical adjustments, we assume all variables are continuous (see [11] for details for adjusting discrete variables vs continuous ones).

Definition 2.1 *Let X be a continuous random variable with cumulative distribution function F. The Extended Gini of X of order k is defined as:*

$$EG_k(X) = E(X) - E\{\min(X_1,...,X_k)\}$$ *, with k a positive integer number*,

where $X, X_1,...,X_k$ are identical independent distributed (i.i.d.) random variables.

The parameter k captures the trader perception of the variability of X. The higher *k*, the more weight $EG_k(X)$ attaches to the left tail[1] of X and, since the right tail of X coincides with the left tail of $-X$, the more weight $EG_k(-X)$ attaches to the right tail of X. As expected, if the distribution of X is symmetric and/or $k=2$, the dispersion on the left and right tail is equal and $EG_k(X) = EG_k(-X)$. Extended Gini of $-X$ is

$$EG_k(-X) = E(-X) - E\{\min(-X_1,...,-X_k)\} = E\{\max(X_1,...,X_k)\} - E(X)$$

It easy to check that $EG_k(X)$ and $EG_k(-X)$ assume non-negative values and, in general,

$$EG_k(X) \neq EG_k(-X).$$

Definition 2.2 $EG_k(X)$ *is called the risk-premium and* $EG_k(-X)$ *the gain-premium of* X *of order* k *.*

We now introduce the notion of bid-price and ask-price as the certainty equivalent of X. The bid-

[1] Note that the definition of ask and bid prices of a firm has been recently introduced by Modesti in [12].

price coincides with the standard definition of certainty equivalent (see [11]); vice versa, that of the ask-price needs to be formulated.

Definition 2.3 *Let the bid-price and the ask-price (or short-selling price) according to the Extended Gini of order k given by*

$$CEG_{bid}^{(k)}(X) = E(X) - EG_k(X) = E\{\min(X_1,...,X_k)\}$$
$$CEG_{ask}^{(k)}(X) = E(X) + EG_k(-X) = E\{\max(X_1,...,X_k)\}$$

where k is a positive integer and $X, X_1,...,X_k$ are i.i.d. random variables.

By definition, the higher k, the higher the risk-premium and the gain-premium, so the lower the bid-price and the higher the ask-price.

Since X is non-negative variable, then $E(X) = E(|X|) = \int_0^\infty (1 - F(t))dt$ and we get

$$CEG_{bid}^{(k)}(X) = \int_0^\infty (1 - F(t))^k dt \text{ and } CEG_{ask}^{(k)}(X) = \int_0^\infty 1 - F^k(t)dt .$$

Then, $CEG_{bid}^{(k)}(X)$ coincides with the Choquet integral with power distortion function $g(t) = t^k$ and $CEG_{ask}^{(k)}(X)$ with its dual-power distortion function $\breve{g}(t) = 1 - (1-t)^k$. Following the terminology used by Chateauneuf *et al.* ([9], Example 4), the parameter k (with $k \geq 1$) is the buyer index of pessimism in $CEG_{bid}^{(k)}(X)$ and the seller index of greediness in $CEG_{ask}^{(k)}(X)$. Clearly, these indices may differ from each other, so it would be necessary to substitute *k* with k_{buyer} and k_{seller}.

3. CLOSED-END FORMULAE FOR BID AND ASK PRICES FOR COMMON STOCHASTIC DISTRIBUTIONS

The bid and ask prices admit closed-end formulae for many familiar distributions used in financial and managerial modelling. They appear in the literature as special cases of more complicated expressions for moments of order statistics. For completeness some derivations are given in the Appendix. In the following, for simplicity of notation, we skip the under-symbol of k_{buyer} and k_{seller} and we write k. To compute the proper $CEG_{bid}^{(k_{buyer})}(X)$ and $CEG_{ask}^{(k_{seller})}(X)$ it is sufficient to substitute the proper values of k_{buyer} and k_{seller} in the formulae.

3.1 Uniform distribution

Let $F_\theta(x) = x/\theta$ for $0 \leq x \leq \theta$, *i.e.,* the uniform distribution on $[0,\theta]$, with mean $\mu = \frac{\theta}{2}$. Due to symmetry, $EG_k(X) = EG_k(-X) = \frac{\theta(k-1)}{2(k+1)}$ and

$$CEG_{bid}^{(k)}(X) = \frac{\theta}{k+1} \text{ and } CEG_{ask}^{(k)}(X) = \frac{k\theta}{k+1}.$$

3.2 Normal distribution

Although $E(\min(X_1,...,X_k))$ and $E(\max(X_1,...,X_k))$ for *i.i.d.* $X_1,...,X_k \sim N(\mu,\sigma^2)$ do exist and recursion formulae can be set up (see [13]), but no closed-form formulae exist even if for small k. For $k = 2$, $EG_2(X) = EG_2(-X) = \frac{\sigma}{\sqrt{\pi}}$ so we get

$$CEG_{bid}^{(2)}(X)=\mu-\frac{\sigma}{\sqrt{\pi}} \text{ and } CEG_{ask}^{(2)}(X)=\mu+\frac{\sigma}{\sqrt{\pi}}.$$

It is also known that $EG_3(X)=EG_3(-X)=1.5\frac{\sigma}{\sqrt{\pi}}$ and $EG_4(X)=EG_4(-X)\approx 1.03\sigma$.

3.3 Skew-Normal distribution

In recent years the skew normal distribution has been successfully used in financial modelling (see Eling et al. [14] for applications) Since no closed-formulae exist for normals, we realistically think that achieve them for skew-normals would be a very hard task. So, again we study the case $k=2$. Let $X\sim SN(\xi,\omega^2,\alpha)$. Since $CEG_{bid}^{(k)}(X)$ and $CEG_{ask}^{(k)}(X)$ are the minimal and maximal order statistics of k copies of *i.i.d.* random variables X , for $k=2$:

$$CEG_{bid}^{(2)}(X)=E(\min(X_1,X_2))=X_{1\,:\,2} \text{ and } CEG_{ask}^{(2)}(X)=E(\max(X_1,X_2))=X_{2\,:\,2}$$

Rearranging a result of Jamalizadeh and Balakrishnan ([15], page 46) and imposing the independence between X_1 and X_2 and we get[2]

$$CEG_{bid}^{(2)}(X)=\xi+\omega\sqrt{\frac{2}{\pi(1+\alpha^2)}}\left(\alpha-\sqrt{\frac{1}{2}}\right).\text{and } CEG_{ask}^{(2)}(X)=\xi+\omega\sqrt{\frac{2}{\pi(1+\alpha^2)}}\left(\alpha+\sqrt{\frac{1}{2}}\right).$$

As expected, $EG_2(X)=EG_2(-X)=\frac{\omega}{\sqrt{\pi(1+\alpha^2)}}$ achieves its maximum value $\frac{\omega}{\sqrt{\pi}}$ for $\alpha=0$ when the distribution is Gaussian; whereas the minimum $\frac{\omega}{\sqrt{2\pi}}$ occurs for $\alpha=\pm 1$ when it is Half-Gaussian.

3.4 Pareto distribution

Pareto distributions are characterized by long and heavy tails modulated by a parameter $\alpha>0$ (and $\alpha>1$ for finite mean). For this property they are commonly used in Finance for modeling extreme events (see Embrechts and Schmidli [16]). Let $F(t)=1-(c/t)^{\alpha}$ for $t>c$ and $\alpha>1$; and $S(t)=1$ for $t<c$ and $S(t)=(c/t)^{\alpha}$ for $t>c$. A random variable with a Pareto distribution can be written as cX , where X has a Pareto distribution with $c=1$. So we consider the special case $c=1$.

Let $S(t)=1$ for $0\le t\le 1$ and $S(t)=t^{-\alpha}$ for $t>1$. Thus

$$CEG_{bid}^{(k)}(X)=\frac{\alpha k}{\alpha k-1} \quad \text{and} \quad CEG_{ask}^{(k)}(X)=\frac{\alpha^k k!}{(\alpha k-1)\cdots(\alpha-1)}.$$

Just to check, for $k=1$, i.e. risk and gain neutrality, we get the mean $\mu=\alpha/(\alpha-1)$.

For $k=2$ we have $CEG_{bid}^{(2)}=2\alpha/(2\alpha-1)$ and $CEG_{ask}^{(2)}=2\alpha^2/[(2\alpha-1)(\alpha-1)]$. This implies

$$EG_2(X)=EG_2(-X)=\alpha/[(2\alpha-1)(\alpha-1)] .$$

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[2]The skew-normal variable X has mean $\xi+\omega\sqrt{\frac{2}{\pi(1+\alpha^2)}}\alpha$ and standard deviation $\sigma=\omega\sqrt{1-\frac{2\alpha^2}{\pi(1+\alpha^2)}}$. If $\alpha=0$, the skew-normal shrinks to the normal with mean $\mu=\xi$ and $\sigma=\omega$.

3.5 Weibull distribution

Let $S(x)=e^{-(x/\lambda)^m}$. Then X has a Weibull distribution with scale parameter $\lambda>0$ and $m>0$. The mean $\mu=\lambda\Gamma\left(1+\frac{1}{m}\right)$. If $m=2$, $\Gamma(3/2)=\sqrt{\pi}/2$. It follows that in the case $m=2$, we have $\mu=\lambda\sqrt{\pi}/2$. For simplicity of the formulae, let $\lambda=2/\sqrt{\pi}$ and $\mu=1$. This is of no practical importance since multiplying λ by the same constant and all of the results generalize to arbitrary λ very easily. Let $S(x)=e^{-\pi x^2/4}$, then $CEG_{bid}^{(k)}(X)=\frac{1}{\sqrt{k}}$ and $CEG_{ask}^{(k)}(X)=\sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}\frac{1}{\sqrt{j}}$.

For the proof see the Appendix. It follows that

$$EG_k(X)=\mu-CEG_{bid}^{(k)}=1-\frac{1}{\sqrt{k}} \text{ and } EG_k(-X)=CEG_{ask}^{(k)}-\mu=\sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}\frac{1}{\sqrt{j}}-1.$$

In the case $k=2$, the formulae reduce to:

$$CEG_{bid}^{(2)}(X)=\frac{1}{\sqrt{2}} \text{ and } CEG_{ask}^{(2)}(X)=2-\frac{1}{\sqrt{2}}\text{; and } EG_2(X)=EG_2(-X)=1-\frac{1}{\sqrt{2}}.$$

3.6 Exponential distribution

If $m=1$ the Weibull distribution collapses into the exponential distribution. Let $F(t)=1-e^{-\lambda t}$ and $S(t)=e^{-\lambda t}$ with the rate parameter $\lambda>0$ and mean $\mu=\frac{1}{\lambda}$ then $CEG_{bid}^{(k)}(X)=\frac{1}{\lambda k}$ and $CEG_{ask}^{(k)}(X)=\frac{1}{\lambda}\sum_{j=1}^{k}\frac{1}{j}$, whereas $EG_k(X)=\frac{1}{\lambda}\left(\frac{k-1}{k}\right)$ and $EG_k(-X)=\frac{1}{\lambda}\sum_{j=2}^{k}\frac{1}{j}$. In the special case $k=2$, $CEG_{bid}^{(2)}=\frac{1}{2\lambda}$ and $CEG_{ask}^{(2)}=\frac{3}{2\lambda}$, and $EG_2(X)=EG_2(-X)=\frac{1}{2\lambda}$.

The results are summarized in Table 1.

		$CEG_{bid}^{(k)}$	$CEG_{ask}^{(k)}$
Uniform (θ)		$\frac{\theta}{k+1}$	$\frac{k\theta}{k+1}$
Normal (μ,σ^2)	$k=2$	$\mu-\frac{\sigma}{\sqrt{\pi}}$	$\mu+\frac{\sigma}{\sqrt{\pi}}$
Skew-Normal SN $(0,\omega^2,\alpha)$	$k=2$	$\xi+\omega\sqrt{\frac{2}{\pi(1+\alpha^2)}}\left(\alpha-\sqrt{\frac{1}{2}}\right)$	$\xi+\omega\sqrt{\frac{2}{\pi(1+\alpha^2)}}\left(\alpha+\sqrt{\frac{1}{2}}\right)$
Pareto $(\alpha,c=1)$		$\frac{\alpha k}{\alpha k-1}$	$\frac{\alpha^k k!}{(\alpha k-1)\cdots(\alpha-1)}$
Weibull $(m=2,\lambda=2/\sqrt{\pi})$		$\frac{1}{\sqrt{k}}$	$\sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}\frac{1}{\sqrt{j}}$
Exponential (λ)		$\frac{1}{\lambda k}$	$\frac{1}{\lambda}\sum_{j=1}^{k}\frac{1}{j}$

TABLE 1. The bid-price $CEG_{bid}^{(k)}$ and the ask-price $CEG_{ask}^{(k)}$.

From a practical point of view, the elicitation of the bid and ask prices needs the knowledge of the personalized orders k capturing the decision maker aversion to risk and propension to gain. A way to achieve these values is to submit to the decision maker some simple tests, see Cardin *et al.* [8].

4. THE TRADING: SUFFICIENT AND NECESSARY CONDITIONS

In the same path of Dow and Da Costa Werlang [17], Chateauneuf and Ventura [10] and Dominiak *et al.* [18], we set up conditions for successful trading. An example will set a stage of our argument.

Example *Consider an asset with random return X. Three actors are in the market: (1) a buy-side investor who declares her bid-price π_{bid}, i.e. the highest price she will pay for X; (2) a sell-side investor declaring her ask-price π_{ask}, i.e. the lowest price she will sell X short; (3) the market-maker. Let the three actors assume the same distribution for X. Successful trading depends on the market price p exogenously given. If p is greater than π_{ask} the sell-side investor feels the market reports a better-than-expected price, and is induced to go short the asset; vice versa, if p is lower than π_{bid} the buy-side investor evaluates the asset to be priced lower-than-expected and tends to go long.*

Let $\pi_{bid} = CEG_{bid}^{(k_{buyer})}(X)$ and $\pi_{ask} = CEG_{ask}^{(k_{seller})}(X)$, then the investor

1) buys the asset X iff $p \le \pi_{bid}$;
2) sells the asset X iff $p \ge \pi_{ask}$;
3) but trade inertia occurs if $\pi_{bid} < p < \pi_{ask}$.

4.1 Probability of trading

A spontaneous question may arise: given a risky asset X how to evaluate the chances of trading? Generally speaking one would think that the bigger the bid ask price spread the greater the chance of trading, but that is not necessarily so. Indeed, the spread is expressed in absolute terms, so it could depend on currency. The smaller the currency unit, the greater the asset value and the greater the spread. Then the spread that does not take into account the distribution of the price itself. We suggest to use as a measure of successful trading just the probability of trading. One can easily check that the probability of no-trade given below does not depend on scaling parameters or translation parameters of the probability distributions involved.
To achieve simple formulae we assume that the both traders assume the same distribution for X and $k_{buyer} = k_{seller} = k$. Clearly, if k_{buyer} differs from k_{seller} it is sufficient to substitute k with the proper value.

Definition 4.2 *If the three actors on the market assume the same continuous distribution function F for X. Then*

$$P(buy\ X) = P(p < \pi_{bid}) = F(\pi_{bid})$$
$$P(sell\ X) = P(p > \pi_{ask}) = 1 - F(\pi_{ask})$$
$$P(trade\ inertia) = P(\pi_{bid} \le p \le \pi_{ask}) = F(\pi_{ask}) - F(\pi_{bid})$$

These probabilities could be computed in many cases using the formulas for CEG_{bid} and CEG_{ask} given in Section 3. Just to grasp a quantitative evaluation on the impact of the distribution asymmetry and k on the probability of buying, selling, and no-trading we compute these quantities and collect them in Table 3-5.

$P(p<\pi_{bid})$	$k=2$	$k=3$	$k=4$
Normal(μ,σ^2)	.2877	.1969	.1539
Skew-Normal SN $\left(0,\omega^2,\alpha\right)$	.3199		
Weibull(m=2, λ)	.3248	.2303	.1783
Uniform(θ)	.3333	.2500	.2000
Exponential (λ)	.3935	.2834	.2212
Pareto ($\alpha=2, c=1$)	.4375	.3056	.2344

TABLE 2. Probability of buying

$P(p>\pi_{ask})$	$k=2$	$k=3$	$k=4$
Pareto ($\alpha=2, c=1$)	.1406	.0976	.0747
Exponential (λ)	.2231	.1599	.1245
Weibull (m=2, λ)	.2690	.1893	.1454
Normal (μ,σ^2)	.2877	.1969	.1539
Skew-Normal SN $\left(0,\omega^2,1\right)$	.3073		
Uniform (θ)	.3333	.2500	.2000

TABLE 3. Probability of selling

$P(\pi_{bid} \le p \le \pi_{ask})$	$k=2$	$k=3$	$k=4$
Uniform (θ)	.3333	.5000	.6000
Skew-Normal SN $\left(0,\omega^2,1\right)$	.3728		
Exponential (λ)	.3834	.5567	.6543
Weibull (m=2, λ)	.4062	.5804	.6763
Pareto ($\alpha=2,\ c=1$)	.4219	.5968	.6909
Normal (μ,σ^2)	.4246	.6062	.6922

TABLE 4. Probability of trade-inertia.

The analysis of data in Tables 2-4 confirms what intuition suggests:

1) The more the probability mass is on the left-tail of X (such as the right-skewed exponential and Pareto distributions), the higher the probability that the market-price is lower than the buyer's bid-price, and consequently the higher the probability of buying. For a negatively-skewed asset the result is reversed.
2) For symmetrical distributions (such as the normal and uniform ones) the probability that the asset being sold or bought is the same.
3) Among the distributions studied, the uniform distribution has the smallest probability of no-trade and the normal distribution has the largest one. This might seem like a reasonable conjecture for all fixed k and all distributions, but the following two examples show that it is false.

Example 4.3 *One up/down jump asset:* $P\left(no-trade\right)=0$ *for all* k *.*

Consider the classical two states asset with one up/down jump in price. Let the asset be represented by the discrete variable X where $P(X=0)=(1-\theta)$ and $P(X=a)=\theta$ with $a>0$ and $0<\theta<1$. Then since $0<E(\min(X_1,...,X_k))\le E(\max(X_1,...,X_k))<a$ for all k, we have $P(p<E(\min(X_1,...,X_k)))=P(p<\pi_{bid})=1-\theta$ for all k. If θ is close to 0, this probability is close to 1 and investors tend to buy. Similarly, we have $P(p>\pi_{ask})=\theta$, so if θ is close to 1, asset-holders tend to sell. We have $P(\pi_{bid} \le p \le \pi_{ask}) = 0$ for all k and all θ. Therefore the probability that this asset is traded is 1.

In conclusion, the one jump asset has probability 1 of being traded for every level of risk-aversion/gain-propension k of the investors.

That perfectly matches with intuition, when we deal with very skewed distributions. If the probability mass is highly concentrated around 0, the left-tail is short, but the right-tail may be very long if $a > 0$ is large. So the buyers tend to buy, since the bargaining seems not too risky, whereas the asset-holders tend not to sell hoping in a high stake. So, a bullish trend is expected. Vice versa, if the probability mass is highly concentrated around a, the left-tail may be very long and the right-one short. So, the risk-averse buyers tend to refuse the bargaining, whereas the asset-holders tend to get rid of the asset. So, the market is prone to undergo a bearish trend.

Of course, this is a discrete distribution. However, it can be approximated to any degree by a continuous distribution[3] which gets highly concentrated around 0 and a and the results would be similar. The probability of trade-inertia might not be exactly 0, but it could be arbitrarily close to 0. There would be other distributions that give 0 probability of no trade for $k = 2$, but no other distribution would have this property for all k since for a distribution on $[a,b]$, we would have $\pi_{bid} \to a$ and $\pi_{ask} \to b$ so $P(\pi_{bid} < X < \pi_{ask})$ could not be 0 for all k.

Example 4.4 *A distribution concentrated around the mean:* P(no-trade) *close to 1 for all* k.
Consider the discrete case, where $P(X = 0) = P(X = 2) = .05$ and $P(X = 1) = .90$. Then $P(\max(X_1, X_2) = 0) = .0025,\ P(\max(X_1, X_2) = 1) = .9000$ and $P(\max(X_1, X_2) = 2) = .0975$. Thus $E(\max(X_1,..., X_k)) = .9000 + 2(.0975) = 1.095$ and by symmetry $E(\min(X_1,..., X_k)) = .905$. It follows $P(E(\min(X_1,..., X_k)) < p < E(\max(X_1,..., X_k))) = P(X = 1) = .9$ which is greater than the value for the normal distribution as given in the Table 1. Clearly, one could make it arbitrarily close to one by making $P(X = 1)$ arbitrarily close to one, making such an asset illiquid.

Again, that perfectly matches with intuition. The difference with the above Example 4.3 is that now the distribution is symmetric and as we will see in Theorem 4.6 in the following Section, the probability to buy and that to sell is equal. If the probability mass is concentrated about the mean, the both risk-premium and the gain-premium shrink to zero. So, π_{bid} and π_{ask} are both close to the mean value, making the asset be similar to a safe asset. That induces investors to have no-interest in bargaining. And, the probability of investor inertia becomes close to one.

Once again, such a distribution could be approximated arbitrarily closely by a continuous distribution, if desired. The continuous distribution would have a large probability of being very close to the expected value 1 and have very small probability of being close to 0 or 2.

4.2 Influence of the skewness

The above discrete examples have highlighted how skewness may influence the probability of trading. We formally prove that the willingness of trading is driven by the monotonicity of the density function and the index k.

Let X and Y be independent random variables with densities $f(x)$ and $g(y)$, respectively. Denote the cumulative distribution function of X by $F(x)$. Then we have

[3]The approximating distribution can be symmetrical if the probability mass lying on the extremes 0 and a is equal, otherwise it is asymmetrical.

Lemma: $P(X \le Y) = E(F(Y))$.

Then we can state the main Theorem of the Section. A sufficient condition such that the probability of buying is higher than the probability of selling or vice versa. Let assume that the price p is i.i.d. distributed as X .

Theorem 4.5 *Assume that* X *has density* $f(x)$ *with* $f' \le 0$ *over the support of the distribution and* $X, p, X_1, ..., X_k$ *are i.i.d. random variables. Then, for all integer* k

$$P(buy\ X) = P(p < \pi_{bid}) \ge \frac{1}{k+1} \ge P(p < \pi_{ask}) = P(sell\ X).$$

The reversed inequalities hold if $f' \ge 0$.

The above has a clear-cut financial interpretation. If the density is decreasing, then the distribution has a short left-tail. Since the risk of losses is small, the buyer is willing to pay more, and that increases the probability of buying.

The uniform, exponential, and Pareto distributions all satisfy the assumptions of the Theorem, since $f' \le 0$. Thus

For $k = 2$: $\quad \mathrm{P}(p < \pi_{bid}) \ge \frac{1}{3} \ge \mathrm{P}(p > \pi_{ask})$.

For $k = 3$: $\quad \mathrm{P}(p < \pi_{bid}) \ge \frac{1}{4} \ge \mathrm{P}(p > \pi_{ask})$.

For $k = 4$: $\quad \mathrm{P}(p < \pi_{bid}) \ge \frac{1}{5} \ge \mathrm{P}(p > \pi_{ask})$.

All of these inequalities are verified by direct calculations collected in Tables 1-3.

Theorem 4.6 *Assume that* X *has a symmetric distribution about its mean* μ *. Then for all integer* k

$$P(buy) = P(sell)$$

In the special case of the uniform distribution the probability to sell, buy, or not trade has a clear-cut formula.

Theorem 4.7 *Let for* $F_\theta(x) = x/\theta$ *for* $0 \le x \le \theta$ *, i.e., the uniform distribution on* $[0, \theta]$ *probabilities. Then*

$$P(buy) = P(sell) = \frac{1}{k+1} \text{ and } P(no-trade) = \frac{k-1}{k+1}.$$

It follows that as the parameter k increases, the probability of trading decreases and that of no-trading increases. That is quite intuitive, since as the buyer risk-aversion increases and the seller gain-propension increases the bid-ask spread increases and the probability of successful trading flaws down.

Above formulas have another fascinating interpretation. Consider the probability that p lies between the min/max of k *i.i.d.* distributed draws of X (note we do no longer consider their expectation, so we do no longer deal with the bid and ask prices):

$$P(\min(X_1, ..., X_k) < p < \max(X_1, ..., X_k)).$$

The following lemma provides its value. Note that is valid for every distribution.

Lemma 4.8 *Let the market price p and* $X_1, ..., X_k$ *is i.d.d. continuous copies of* X *. Then*

$$P(\min(X_1,...,X_k) < p < \max(X_1,...,X_k)) = \frac{k-1}{k+1}.$$

Above has an immediate interpretation. Suppose k independent experts be asked to express their evaluation on X. They deal in a complete market, so their evaluations X_i, $i=1,..,k$ and p are *i.i.d.* variables. The probability that the market price p be bound by the minimum and the maximum variables expressed by the experts is given by $\frac{k-1}{k+1}$. So, if $k=2$, the probability is 0.33 and it increases till 1 if the number k of experts increases to infinity.

5. CONCLUSION

Bid and ask prices tailored to individual trader's attitudes are defined through the Extended Gini premium principle. The buyer risk-aversion and the seller gain-propension are captured by personalized parameters. On the path of Chateauneuf and Ventura [10], we state sufficient and necessary conditions for successful trading. We use the probability of trading for measuring the chances of successful trading. Closed-end formulae and tables for these measures for the most common distributions used in risk management (uniform, normal, skew-normal, Pareto, Weibull, exponential distribution) are given. A set of guidelines is set up: (1) if the asset probability mass is decreasing from the left-tail of X to the right-one, the probability of buying exceeds that of selling and vice versa (see Theorem 4.5); (2) if the asset distribution is symmetrical then the probability of buying is equal of that of selling (see Theorem 4.6); (3) among the distributions studied, the uniform distribution has the smallest probability of no-trade although it increases as k increases and the normal distribution has the largest one. However, there exist distributions where the probability of the investor inertia is arbitrarily close to 0 or 1 for all k.

A further interesting aspect to investigate is whether the above normative pricing method matches real market exchange data, but that question is left to future research.

6. APPENDIX

Sec. 3.4 (Pareto)

Let $CEG_{bid}^{(k)} = E(\min(X_1,..,X_n)) = \int_0^1 1dt + \int_1^\infty t^{-\alpha k}dt = 1 + \frac{1}{\alpha k - 1} = \frac{\alpha k}{\alpha k - 1}$

Vice versa, the density of $\max(X_1,...,X_k)$ is $k\alpha(1-t^{-\alpha})^{k-1}t^{-\alpha-1}$. Thus

$$CEG_{ask}^{(k)}(X) = E(\max(X_1,..,X_k)) = \int_1^\infty k\alpha t(1-t^{-\alpha})^{k-1}t^{-\alpha-1}dt = \int_0^1 k(1-u)^{k-1}u^{-1/\alpha}du = \frac{k\Gamma(k)\Gamma(1-1/\alpha)}{\Gamma(k+1-1/\alpha)} = \frac{k!}{\left(k-\frac{1}{\alpha}\right)\cdots\left(1-\frac{1}{\alpha}\right)}$$

.

Sec. 3.5 (Weibull)

Since $P(\min(X_1,...,X_k) > x) = e^{-\pi kx^2/4}$, it follows that $E(\min(X_1,...,X_k)) = \frac{1}{\sqrt{k}}$. Then, since

$$E(\max(X_1,...,X_k)) = \int_0^\infty 1-(1-e^{-\pi x^2/4})^k dx = \int_0^\infty \sum_{j=1}^k \binom{k}{j}(-1)^{j+1}e^{-\pi jx^2/4}dx = \sum_{j=1}^k (-1)^{j+1}\binom{k}{j}\frac{1}{\sqrt{j}}$$

and it proves our thesis.

Sec. 3.6 (Exponential)

Let $CEG_{bid}^{(k)} = \int_0^\infty e^{-\lambda kx}dx = \frac{1}{\lambda k}$ so $EG_k(X) = \frac{k-1}{\lambda k}$. Then $CEG_{ask}^{(k)} = \int_0^\infty 1-(1-e^{-\lambda x})^k dx = \frac{1}{\lambda}\sum_{j=1}^k \frac{1}{j}$.

Proof of Lemma in Sec. 4.2

$P(X \le Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{y} f(x)g(y)dxdy = \int_{-\infty}^{\infty} F(y)g(y)dy = \mathrm{E}(F(Y))$.

Proof of Theorem 4.5

It follows from the assumption that $F'' \le 0$ so F is a concave function. Thus from Jensen's inequality (for concave, not convex functions), we have

$$F\left(E\left(\min\left(X_1,..,X_k\right)\right)\right) \ge E\left(F\left(\min\left(X_1,..,X_k\right)\right)\right),$$

where the left side is just $P(p \ge \mathrm{E}(\max(X_1,...,X_k))$ and by the Lemma, the right side is $P(p \le \mathrm{E}(\min(X_1,...,X_k))$. Since $p, X_1,...,X_k$ are *i.i.d.* continuous random variables, this last probability is just $1/(k+1)$ by symmetry. We thus have proved the inequality.

To prove the second inequality we note that $S(x) = 1 - F(x)$ is a convex function. Thus from Jensen's inequality, we have

$$S\left(E\left(\max\left(X_1,..,X_k\right)\right)\right) \le E\left(S\left(\max\left(X_1,..,X_k\right)\right)\right),$$

where the left side is just $\mathrm{P}(p \ge (\max(X_1,...,X_k))$ and the right side is $\mathrm{E}\left(\mathrm{P}(p \ge (\max(X_1,...,X_k))\right)$. Since $p, X_1,...,X_k$ are *i.i.d.* continuous random variables, this last probability is just $1/(k+1)$ by symmetry. We thus have thus proved the second inequality.

Vice versa, if $f' \ge 0$ F will be convex and S will be concave and the reversed inequalities hold.

Proof of Theorem 4.6

For simplicity assume first that X is symmetric about 0. Then X has the same distribution as $-X$. Thus $\max(X_1,...,X_k)$ has the same distribution as $\max(-X_1,...,-X_k) = -\min(X_1,...,X_k)$. Hence $\mathrm{E}(\max(X_1,...,X_k) = -\mathrm{E}(\min(X_1,..,X_k))$. Since p is symmetric, $P(p < -a) = P(p > a)$ for all α. In particular,

$P\left(p < E\left(\min\left(X_1,..,X_k\right)\right)\right) = P\left(p < -E\left(\max\left(X_1,..,X_k\right)\right)\right) = \left(p > E\left(\max\left(X_1,..,X_k\right)\right)\right)$.

If symmetric about μ, then $X - \mu$ is symmetric about 0, so

$$P(p-\mu < \mathrm{E}(\min(X_1-\mu,...,X_k-\mu)) = P(p-\mu > \mathrm{E}(\max(X_1-\mu,...,X_k-\mu)).$$

Thus

$$P(p-\mu < \mathrm{E}(\min(\left(X_1,...,X_k\right)-\mu)) = P(p-\mu > \mathrm{E}(\max\left(X_1,...,X_k\right)-\mu)).$$

Adding μ, it follows that

$$P(p < \mathrm{E}(\min(X_1,...,X_k)) = P(p > \mathrm{E}(\max(X_1,...,X_k)).$$

Proof of Lemma 4.8

By continuity of the random variables, we may assume that exactly one random variable takes the maximum value and one random variable takes the minimum value. The complement of the event $A = P(\min(X_1,...,X_k) < p < \max(X_1,...,X_k))$ is thus the union of the events $B = \{p = \min(X_1,...,X_k,p)\}$ and $C = \{p = \max(X_1,...,X_k,p)\}$. By symmetry, each random variable has an equal probability of being the maximum or minimum of $\{X_1,...,X_k,p\}$. Thus $P(B) = P(C) = \frac{1}{k+1}$ and $P(A) = 1 - \frac{2}{k+1} = \frac{k-1}{k+1}$.

Proof of Theorem 4.7

Since $E(\min(X_1,...,X_k)) = \frac{\theta}{k+1},\ E(\max(X_1,...,X_k)) = \frac{k\theta}{k+1}$, it follows

$$P\left(\pi_{bid} < p < \pi_{ask}\right) = \frac{k-1}{k+1}$$

That is the probability given in the Lemma 4.8. Since the distribution is symmetric

$$P\left(p<\pi_{bid}\right)=P\left(p>\pi_{ask}\right)=\frac{1}{2}\left(1-\frac{k-1}{k+1}\right)=\frac{1}{k+1}.$$

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