Embedding and Np-Complete Problems for 3-Equitable Graphs

S. K. Vaidya samirkvaidya@yahoo.co.in
Department of Mathematics,
Saurashtra University,
RAJKOT – 360005, Gujarat(India).

P. L. Vihol viholprakash@yahoo.com
Department of Mathematics,
Government Engineering College,
RAJKOT – 360003, Gujarat(India).

Abstract

We present here some important results in connection with 3-equitable graphs. We prove that any graph G can be embedded as an induced subgraph of a 3-equitable graph. We have also discussed some properties which are invariant under embedding. This work rules out any possibility of obtaining any forbidden subgraph characterization for 3-equitable graphs.

Keywords: Embedding, NP-Complete, 3-Equitable Graph.

2000 Mathematics Subject Classification: 05C78

1. INTRODUCTION

We begin with simple, finite, connected and undirected graph $G=(V(G),E(G))$, where $V(G)$ is called set of vertices and $E(G)$ is called set of edges of a graph G. For all other terminology and notations in graph theory we follow West [1] and for number theory we follow Niven and Zuckerman [2].

Definition 1.1 The assignment of numbers to the vertices of a graph with certain condition(s) is called graph labeling.

For detailed survey on graph labeling we refer to Gallian [3]. Vast amount of literature is available on different types of graph labeling and more than 1200 papers have been published in past four decades. As stated in Beineke and Hegde [4] graph labeling serves as a frontier between number theory and structure of graphs. Most of the graph labeling techniques trace there origin to that one introduced by Rosa [5].

Definition 1.2

Let $G=(V(G),E(G))$ be a graph with p vertices and q edges. Let $f:V\rightarrow\{0,1,2,\ldots,q\}$ be an injection. For each edge $uv\in E$, define $f^*(uv)=|f(u)-f(v)|$. If $f^*(E)=\{1,2,\ldots,q\}$ then f is called β-valuation. Golomb [6] called such labeling as a graceful labeling and this is now the familiar term.

Definition 1.3

For a mapping $f:V(G)\rightarrow\{0,1,2,...k-1\}$ and an edge $e=uv$ of G, we define $f(e)=|f(u)-f(v)|$. The labeling f is called a k - equitable labeling if the number of vertices with the label i and the number of vertices with the label j differ by atmost 1 and the number of edges with the label i and the number of edges with label j differ by atmost 1. By $v_f(i)$ we mean the number of vertices with the label i and by $e_f(i)$ we mean the number of edges with the label i.

Thus for k - equaitable labeling we must have $|v_f(i)-v_f(j)|\le 1$ and $|e_f(i)-e_f(j)|\le 1$, where $0\le i,j\le k-1$.

For $k=2$, f is called cordial labeling and for $k=3$, f is called 3-equitable labeling. We focus on 3-equitable labeling.

A graph G is *3-equitable* if it admits a 3-equitable labeling. This concept was introduced by Cahit [7]. There are four types of problems that can be considered in this area.

(1) How 3-equatability is affected under various graph operations.
(2) Construct new families of 3-equitable graphs by finding suitable labeling.
(3) Given a graph theoretic property P characterize the class of graphs with property P that are 3-equitable.
(4) Given a graph G having the graph theoretic property P, is it possible to embed G as an induced subgraph of a 3-equitable graph G, having the property P ?

The problems of first three types are largely investigated but the problems of last type are of great importance. Such problems are extensively explored recently by Acharya et al [8] in the context of graceful graphs. We present here an affirmative answer for planar graphs, trianglefree graphs and graphs with given chromatic number in the context of 3-equitable graphs. As a consequence we deduce that deciding whether the chromatic number is less then or equal to k, where $k\ge 3$, is NP-complete even for 3-equitable graphs. We obtain similar result for clique number also.

2. Main Results

Theorem 2.1

Any graph G can be embedded as an induced subgraph of a 3-equitable graph.

Proof: Let G be the graph with n vertices. Without loss of generality we assume that it is always possible to label the vertices of any graph G such that the vertex conditions for 3-equitable graphs are satisfied. *i.e.* $|v_f(i)-v_f(j)|\le 1,\ 0\le i,j\le 2$. Let V_0, V_1 and V_2 be the set of vertices with label 0, 1 and 2 respectively. Let E_0, E_1 and E_2 be the set of edges with label 0,1 and 2 respectively. Let $n(V_0)$, $n(V_1)$ and $n(V_2)$ be the number of elements in sets V_0, V_1 and V_2 respectively. Let $n(E_0)$, $n(E_1)$ and $n(E_2)$ be the number of elements in sets E_0, E_1 and E_2 respectively.

<u>**Case 1:**</u> $n\equiv 0(mod\quad 3)$

<u>**Subcase 1:**</u> $n(E_0)\ne n(E_1)\ne n(E_2)$.

Suppose $n(E_0)<n(E_1)<n(E_2)$. Let $|n(E_2)-n(E_0)|=r>1$ and $|n(E_2)-n(E_1)|=s>1$. The new graph H can be obtained by adding $r+s$ vertices to the graph G.

Define $r+s=p$ and consider a partition of p as $p=a+b+c$ with $|a-b|\le 1, |b-c|\le 1$ and $|c-a|\le 1$.

Now out of new p vertices label a vertices with 0, b vertices with 1 and c vertices with 2.i.e. label the vertices $u_1,u_2,\dots,u_a$ with 0, $v_1,v_2,\dots,v_b$ with 1 and $w_1,w_2,\dots,w_c$ with 2.Now we adapt the following procedure.

<u>**Step 1:**</u> To obtain required number of edges with label 1.

- Join s number of elements v_i to the arbitrary element of V_0.
- If $b<s$ then join ($s-b$) number of elements $u_1,u_2,\dots,u_{s-b}$ to the arbitrary element of V_1.
- If $a<s-b$ then join ($s-a-b$) number of vertices $w_1,w_2,\dots,w_{s-b-a}$ to the arbitrary element of V_1.

Above construction will give rise to required number of edges with label 1.

<u>**Step 2:**</u> To obtain required number of edges with label 0.

- Join remaining number of u_i 's (which are left at the end of step 1) to the arbitrary element of V_0.

• Join the remaining number of v_i's(which are left at the end of step 1) to the arbitrary element of V_1.

• Join the remaining number of w_i's(which are left at the end of step 1) to the arbitrary element of V_2.

As a result of above procedure we have the following vertex conditions and edge conditions.

$$|v_f(0)-v_f(1)|=|n(V_0)+a-n(V_1)-b|\leq 1,$$

$$|v_f(1)-v_f(2)|=|n(V_1)+b-n(V_2)-c|\leq 1,$$

$$|v_f(2)-v_f(0)|=|n(V_2)+c-n(V_0)-a|\leq 1$$

and

$$|e_f(0)-e_f(1)|=|n(E_0)+n(E_2)-n(E_0)-n(E_1)-n(E_2)+n(E_1)|=0,$$

$$|e_f(1)-e_f(2)|=|n(E_1)+n(E_2)-n(E_1)-n(E_2)|=0,$$

$$|e_f(2)-e_f(0)|=|n(E_2)-n(E_0)-n(E_2)+n(E_0)|=0.$$

Similarly one can handle the following cases.

$n(E_0)<n(E_2)<n(E_1)$,

$n(E_2)<n(E_0)<n(E_1)$,

$n(E_1)<n(E_2)<n(E_0)$,

$n(E_2)<n(E_1)<n(E_0)$,

$n(E_1)<n(E_0)<n(E_2)$.

Subcase 2: $n(E_i)=n(E_j)<n(E_k), i\neq j\neq k, 0\leq i,j,k\leq 2$

Suppose $n(E_0)=n(E_1)<n(E_2)$

$$|n(E_2)-n(E_0)|=r$$

$$|n(E_2)-n(E_1)|=r$$

The new graph H can be obtained by adding $2r$ vertices to the graph G.

Define $2r=p$ and consider a partition of p as $p=a+b+c$ with $|a-b|\leq 1, |b-c|\leq 1$ and $|c-a|\leq 1$.

Now out of new p vertices, label a vertices with 0, b vertices with 1 and c vertices with 2.i.e. label the vertices $u_1, u_2, \ldots, u_a$ with 0, $v_1, v_2, \ldots, v_b$ with 1 and $w_1, w_2, \ldots, w_c$ with 2.Now we adapt the following procedure.

Step 1:

To obtain required number of edges with label 0.

• Join r number of elements $u_i's$ to the arbitrary element of V_0.

• If $a<r$ then join ($r-a$) number of elements $v_1, v_2, \ldots, v_{r-a}$ to the arbitrary element of V_1.

• If $b<r-a$ then join ($r-a-b$) number of vertices $w_1, w_2, \ldots, w_{r-b-a}$ to the arbitrary element of V_2.

Above construction will give rise to required number of edges with label 0.

Step 2:

To obtain required number of edges with label 1.

• Join remaining number of w_i's (which are not used at the end of step 1)to the arbitrary element of V_1.

• Join the remaining number of $v_i's$ (which are not used at the end of step 1) to the arbitrary element of V_0.

• Join the remaining number of $u_i's$ (which are not used at the end of step 1) to the arbitrary element

of V_1.
Similarly we can handle the following possibilities.
$n(E_1) = n(E_2) < n(E_0)$
$n(E_0) = n(E_2) < n(E_1)$

Subcase 3 : $n(E_i) < n(E_j) = n(E_k), i \neq j \neq k, 0 \leq i, j, k \leq 2$

Suppose $n(E_2) < n(E_0) = n(E_1)$

Define $| n(E_2) - n(E_0) | = r$

The new graph H can be obtained by adding r vertices to the graph G as follows .
Consider a partition of r as $r = a + b + c$ with $| a - b | \leq 1, | b - c | \leq 1$ and $| c - a | \leq 1$.

Now out of new r vertices label a vertices with 0, b vertices with 1 and c vertices with 2 .i.e. label the vertices $u_1, u_2, \ldots, u_a$ with 0, $v_1, v_2, \ldots, v_b$ with 1 and $w_1, w_2, \ldots, w_c$ with 2. Now we adapt the following procedure.

Step 1:
To obtain required number of edges with label 2.

- Join r number of vertices $w_i's$ to the arbitrary element of V_0.
- If $c < r$ then join $r - c$ number of elements $u_1, u_2, \ldots, u_{r-c}$ to the arbitrary element of V_2.

Above construction will give rise to required number of edges with label 2.
At the end of this step if the required number of 2 as edge labels are generated then we have done. If not then move to step 2. This procedure should be followed in all the situations described earlier when $n(E_2) < n(E_0)$ or $n(E_2) < n(E_1)$.

Step 2:
To obtain the remaining (at the end of step 1) number of edges with label 2.

- If k number of edges are required after joining all the vertices with label 0 and 2 then add k number of vertices labeled with 0, k number of vertices labeled with 1 and k number of vertices labeled with 2. Then vertex conditions are satisfied.
- Now we have k number of new vertices with label 2, k number of new vertices with label 0 and $2k$ number of new vertices with label 1.
- Join k new vertices with label 2 to the arbitrary element of the set V_0.
- Join k new vertices with label 0 to the arbitrary element of the set V_2.
- Join k new vertices with label 1 to the arbitrary element of set V_0.
- Join k new vertices with label 1 to the arbitrary element of the set V_1.

Case 2: $n \equiv 1 (mod 3)$.

Subcase 1: $n(E_i) \neq n(E_j) \neq n(E_k), i \neq j \neq k, 0 \leq i, j, k \leq 2$.

Suppose $n(E_0) < n(E_1) < n(E_2)$ Let $| n(E_2) - n(E_0) | = r > 1$ and $| n(E_2) - n(E_1) | = s > 1$.

Define $r + s = p$ and consider a partition of p such that $p = a + b + c$ with

$$| n(V_0) + a - n(V_1) - b | \leq 1$$
$$| n(V_1) + b - n(V_2) - c | \leq 1$$
$$| n(V_0) + a - n(V_2) - c | \leq 1.$$

Now we can follow the procedure which we have discussed in case-1.

Case 3: $n \equiv 2 (mod 3)$

We can proceed as case-1 and case-2.
Thus in all the possibilities the graph H resulted due to above construction satisfies the conditions for 3-equitable graph. That is, any graph G can be embedded as an induced subgraph of a 3-

equitable graph.
For the better understanding of result derived above consider following illustrations.

Illustration 2.2

For a Graph $G = C_9$ we have $n(E_0) = 0$, $n(E_1) = 6$, $n(E_2) = 3$.

Now $|n(E_1) - n(E_0)| = 6 = r$, $|n(E_1) - n(E_2)| = 3 = s$.

This is the case related to subcase (1) of case (1).

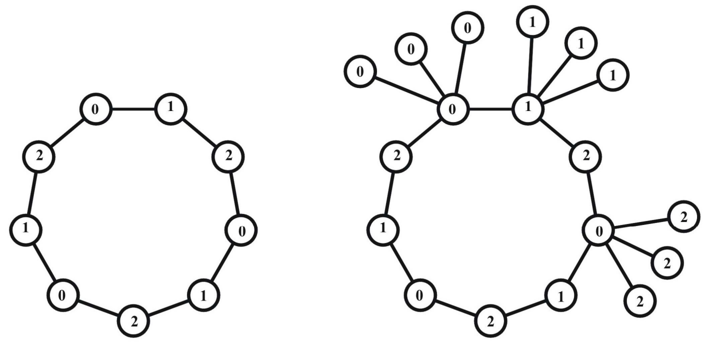

FIGURE 1: C_9 and its 3-equitable embedding

Procedure to construct H :

Step 1:

- Add $p = r + s = 6 + 3 = 9$ vertices in G and partition p as $p = a + b + c = 3 + 3 + 3$.
- Label 3 vertices with 0 as $a = 3$.
- Label 3 vertices with 1 as $b = 3$.
- Label 3 vertices with 2 as $c = 3$.

Step 2:

- Join the vertices with 0 and 1 to the arbitrary element of the set V_0 and V_1 respectively.
- Join the vertices with label 2 to the arbitrary element of set V_0.

The resultant graph H is shown in Figure 1 is 3-equitable.

Illustration 2.3

Consider a Graph $G = K_4$ as shown in following Figure 2 for which $n(E_0) = 1$, $n(E_1) = 4$, $n(E_2) = 1$.

Here $|n(E_1) - n(E_0)| = 3 = r$, $|n(E_1) - n(E_2)| = 3 = s$ i.e. $r = s$.

This is the case related to subcase (2) of case (2).

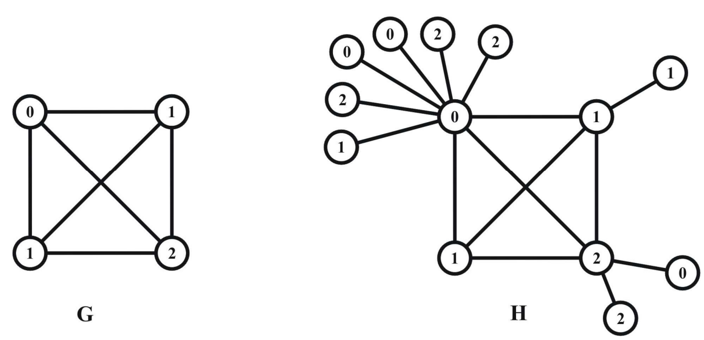

FIGURE 2: K_4 and its 3-equitable embedding

Procedure to construct H :

Step 1:

- Add $p = 2r = 3 + 3 = 6$ vertices in G and partition p as $p = a + b + c = 2 + 1 + 3$.
- Label 2 vertices with 0 as $a = 2$.
- Label 1 vertex with 1 as $b = 1$.
- Label 3 vertices with 2 as $c = 2$.

Step 2:

- Join the vertices with label 0 to the arbitrary element of the set V_0 and join one vertex with label 2 to the arbitrary element of V_2.
- join the remaining vertices with label 2 with the arbitrary element of set V_0.

Step 3:

- Now add three more vertices and label them as $0, 1$ and 2 respectively.
- Now join the vertices with label 0 and 2 with the arbitrary elements of V_2 and V_0 respectively.
- Now out of the remaining two vertices with label 1 join one vertex with arbitrary element of set V_0

and the other with the arbitrary element of set V_1.

The resultant graph H shown in Figure 2 is 3-equitable.

Corollary 2.4 Any planar graph G can be embedded as an induced subgraph of a planar 3-equitable graph.

Proof: If G is planar graph. Then the graph H obtained by Theorem 2.1 is a planar graph.

Corollary 2.5 Any triangle-free graph G can be embedded as an induced subgraph of a triangle free 3-equitable graph.

Proof: If G is triangle-free graph. Then the graph H obtained by Theorem 2.1 is a triangle-free graph.

Corollary 2.6 The problem of deciding whether the chromatic number $\chi \leq k$, where $k \geq 3$ is NP-complete even for 3-equitable graphs.

Proof: Let G be a graph with chromatic number $\chi(G) \geq 3$. Let H be the 3-equitable graph constructed in Theorem 2.1, which contains G as an induced subgraph.Since H is constructed by adding only pendant vertices to G. We have $\chi(H) = \chi(G)$. Since the problem of deciding whether the chromatic number $\chi \leq k$, where $k \geq 3$ is NP-complete [9]. It follows that deciding whether the chromatic number $\chi \leq k$, where $k \geq 3$, is NP-complete even for 3-equitable graphs.

Corollary 2.7 The problem of deciding whether the clique number $\omega(G) \geq k$ is NP-complete even when restricted to 3-equitable graphs.

Proof: Since the problem of deciding whether the clique number of a graph $\omega(G) \geq k$ is NP-complete [9] and $\omega(H) = \omega(G)$ for the 3-equitable graph H constructed in Theorem 2.1,the above result follows.

3. Concluding Remarks

In this paper, we have considered the general problem. Given a graph theoretic property P and a graph G having P, is it possible to embed G as an induced subgraph of a 3-equitable graph H having the property P ? As a consequence we derive that deciding whether the chromatic number $\chi \leq k$, where $k \geq 3$, is NP-complete even for 3-equitable graphs. We obtain similar result for clique number. Moreover this work rules out any possibility of forbidden subgraph characterization for 3-equitable graph. Analogous work for other graph theoretic parameters like domination number, total domination number, fractional domination number etc. and graphs admitting various other types of labeling can be carried out for further research.

REFERENCES

[1] D B West. Introduction To Graph Theory, Prentice-Hall of India, 2001

[2] I Niven and H Zuckerman. An Introduction to the Theory of numbers, Wiley Eastern, New Delhi, 1972.

[3] J A Gallian. A dynamic survey of graph labeling, The Electronics Journal of Combinatorics, 17, $\#DS6$, 2010.

[4] L W Beineke and S M Hegde. "Strongly Multiplicative Graphs", Discuss.Math.Graph Theory, 21, pp. 63-75, 2001.

[5] A Rosa. "On Certain Valuation of the Vertices of a Graph", Theory of Graphs (Internat. Symposium, Rome, July 1966) Gordon and Breach, N.Y. and Dunod Paris, pp. 349-355, 1967.

[6] S W Golomb. "How to Number a Graph" in: Graph Theory and Computing, R.C.Read, (Eds.), Academic Press, New York, pp. 23-37, 1972.

[7] I Cahit. "On Cordial and 3-equitable Labeling of Graphs",Utilitas.Math, 37, pp. 189-198, 1990.

[8] B D Acharya. S B Rao, S Arumugam, "Embeddings and NP-Complete problems for Graceful Graphs" in: B D Acharya, S Arumugam, A Rosa (Eds.),Labeling of Discrete Structures and Applications, Narosa Publishing House, New Delhi, pp. 57-62, 2008.

[9] M R Garey and D S Johnson. Computers and Intractability- A guide to the theory of NP-Completeness, W.H.Freeman and Company, 1979.