A Stochastic Iteration Method for A Class of Monotone Variational Inequalities In Hilbert Space

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Abstract

We examined a general method for obtaining a solution to a class of monotone variational inequalities in Hilbert space. $Let\ H\ be\ a\ real\ Hilbert\ space, and\ Let\ T: H \to H$ be a continuous linear monotone operator and K be a non empty closed convex subset of H. From an initial arbitrary point $x_0 \in K$. We proposed and obtained iterative method that converges in norm to a solution of the class of monotone variational inequalities. A stochastic scheme$\{x_n\}$ is defined as follows: $x_{n+1} = x_n - a_n F^*(x_n),\ n \geq 0, F^*(x_n), n \geq 0$ is a strong stochastic approximation of $Tx_n - b, for\ all\ b\ (possible\ zero) \in H\ and\ a_n \in (0,1)$.

Key words: Linear Monotone Operator, Hilbert Space, Stochastic Approximation.

1. INTRODUCTION

Many researchers over the years has studied the monotone variational inequalities as a result of its relevance to the many disciplines such partial differential equations, optimization, optimal control and finance.

We will assume that H is Hilbert space with a complete inner product<·,·> with associated induced norm‖·‖.

Let C be a nonempty, closed and convex subset of H. Let the strong convergence and weak convergence of $\{x_n\}\ to\ x \in H\ by\ x_n \to x\ and\ x_n \to x$.

A map $T: H \to H\ is\ said\ to\ be\ monotone\ if\ for\ all\ x, y \in D(T)$

$$\langle Tx - Ty, x - y \rangle \geq 0 \tag{1}$$

$D(T) \in T and\ \beta - strongly\ monotone\ if\ there\ exists\ \beta > 0\ such\ that$

$$\langle Tx - Ty, x - y \rangle \geq \beta \|x - y\|^2,\ for\ all\ x, y \in D(T) \tag{2}$$

If there exists an increasing function

$$\Lambda: (0, \infty) \to R,\ \Lambda(0) = 0 \quad such\ that\ \langle Tx - Ty,\ x - y \rangle \geq\ \Lambda(\|x - y\|) \tag{3}$$

$for\ all\ x, y \in D(T)$ it implies that T is uniformly convex.

Consider the continuous linear monotone operator

$$T: H \to H\ , and\ the\ equation\ Tx = b \tag{4}$$

given on a closed convex subset $K \subset D(T) \subseteq H, b\ can\ be\ possibly\ zero\ in\ H$.

The importance of the solution of (4) is its application in many areas where problems are reduced to determining solutions of equilibrium problems.

In the case of where $b = 0$, it reduces to the fixed point problems $x^* = P_k(x^* - sTx^*)$ where P_k is the nearest point projection map from H $onto$ K for some arbitrary positive fixed constant s.

2. PRELIMINARIES

In the work of Zhou H., Yu Zhou and Guanghui Feng, (see[11]), the authors examined the general regularization method for seeking a solution to a class of monotone variational inequalities in Hilbert space, where the regularizer is a hemi continuous and strongly monotone operator.

Again, in the work of Lud, X. and Yang, J. (see [12]), the authors consider the monotone inverse variational inequalities in a nonempty closed convex subset of a real Hilbert space. The authors showed a general regularization method for monotone inverse variational inequalities, where the regularizer is a Lipschitz continuous and strongly monotone mapping which converges strongly to a solution of the inverse variational inequality.

Our interest in this paper is to examine and fully present a stochastic iteration process which converges strongly to the solution of (4).

Recall that equilibrium problems can be formulated as co-variational inequalities problem involving a continuous linear monotone map.

Find a point $x^* \in K \; such \; that \; CVI(T, K): \langle Tx^* - b, y - x^* \rangle \;\; \forall y \in K$ (5)

The basic relationship existing between (4) and (5) is given by

Theorem 1: $Let \; H \; be \; a \; Hilbert \; space \;\; and \; T: H \to H$ a linear continuous monotone operator, the following statements are equivalent.

(i) x^* satisfies the co-variational inequalities $\langle Tx^* - b, y - x^* \rangle, 0 \; \forall \; y \in H$

(ii) $Tx^* = b$

It is shown that if T is continuous and strongly monotone, then

$CVI(T, K): \langle Tx^* - b, y - x^* \rangle \;\; \forall \; y \in K$ has a unique solution (see [1,2,3]).

Mann, Ishikawa and other iterative processes, have been traditionally devoted to the case where T possesses strong monotonicity properties. A case when T is not necessarily uniform or strongly monotone, some of the known applied iterative schemes such as Mann and Ishikawa iterative processes do not generate a unique solution.

For lots of practical solutions, we are interested to obtain an approximation of x^* from experimental observations.

The introduction of uncertainty provides an accurate method of investigation which suggested the stochastic approximation method.

This work is an extension to infinite-dimensional Hilbert space of the work in the Euclidean space (see [4]).

A new formulation of the operator equation as a new uniquely solvable gradient operator equation, which is approximated using a least squares approximation method.

Let H be a real separable Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and $norm$ $\|\cdot\|$. Given a probability space (Ω, f, P) , a random vector in H is a measurable mapping as defined on the probability space with values in H.

It is a known fact that the following equalities hold.

$$\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2 \;\; for\ \forall x, y \in H \tag{6}$$

Recall also, that the metric projection from a real Hilbert space H onto a nonempty closed, and convex subset $C\ of\ H$ is defined as follows:

For each $x \in H$ there exists a unique element $P_c x \in C$ with the property

$$\|x - P_c x\| \leq \|x - y\|, \forall\ x \in C \tag{7}$$

$$for\ any\ \ x \in H\ , \hat{x} = P_c x \ \ iff$$

$$\hat{x} \in C\ and\ \|x - \bar{x}\| = inf\{\|x - y\| : y \in C\} \tag{8}$$

LEMMA 1: Let P_c be the metric project from H onto a nonempty, bounded, closed and convex subset C of a real Hilbert space H.

$Then\ z = P_c x\ \ iff\ the\ inequality\ \langle x - z,\ \ y - z\rangle \leq 0\ \forall y \in C$ and $x \in H$ (9)

(1) $\|P_c x - P_c y\|^2 \leq \ \langle P_c x - P_c y\ , x - y\rangle,\ \forall x, y \in H$ (10)

An operator $F: C \rightarrow H$ is said to be

(a) Monotone if $\langle Fx - Fy, x - y\rangle \geq 0\ \ \forall x, y \in C$ (11)

(b) $\beta - strongly\ monotone\ if\ there\ exists\ an\ \beta > 0\ such\ that$

$$\langle Fx - Fy, x - y\rangle \geq \beta\|x - y\|^2 \ \ \forall x, y \in C \tag{12}$$

3. FORMULATION OF STOCHASTIC APPLICATION ALGORITHM

Let E be the expectation operator. If $E\|u\| < \infty$, then $E\langle a, u\rangle = \langle a, Eu\rangle\ \forall\ a \in H$

Given any linear continuous monotone map $T: H \rightarrow H$, there exists a continuous, convex and everywhere Frechet differentiable scalar function $f: H \rightarrow R$, such that

$$f(x) = \frac{1}{2}\langle Tx, x\rangle - \langle b. x\rangle \tag{13}$$

The gradient mapping ∇f, uniquely determined for a given $T\ and\ b$ coincides with

$Tx^* - b\ \forall x$. Finding x^* such that $Tx^* = b\ and\ \langle Tx^* - b, y - x^*\rangle \geq 0\ \forall y \in K$ [7],

is equivalent to finding the unique zero $x^*\ of\ \nabla f(x^*)$ such that $0 = \nabla f(x^*)$ (14)

Given that ∇f is a monotone operator defined on Hilbert space into itself then

the iterative scheme

$$x^{k+1} = x^k - a_k \nabla f(x^k) \tag{15}$$

converges strongly to a zero of ∇f for a suitable $\{a_k\}$.

We consider a stochastic approximation algorithm. The usual form of stochastic approximation algorithm is a minimum point $x^* \in H$, of a function $f: H \to R$ is of the form

$$x^{k+1} = x^k - a_k d^k \tag{16}$$

Where $\nabla f(x^k) = \int d(\theta) P_{x_k}(dt)\ and\ P_{x_k}$ is the usual probability law of $d(x^k) = d^k$, d^k is approximation of the gradient $\nabla f(x^k)$, and $\{a_k\}$ is a sequence of positive scalars that decreases to zero (see [5], [6] and [8]).

The important part of (16) is the gradient approximation d^k. We applied objective function measurements and requires the minimization of a continuous and convex function.

Given a sequence of random vectors d^k, computed from different data points:

$P_j \in H$, for each $k\ and\ j = 1, \dots, N$ such that d^k approximates $\nabla f(x^k)$ strongly and in mean square in the sense that the estimated vectors are minimized as follows:

Definition 1: The sequence of random vectors d^k , $\nabla f(x^k)$ is strongly approximates if

$$E\|d^k - \nabla f(x^k)\| = 0\ \ for\ each\ k \tag{17}$$

and consistent with $\nabla f(x^k)$ in mean square if:

$$E\|d^k - \nabla f(x^k)\|^2 \to 0\ as\ N \to \infty \tag{18}$$

where $N \in (x_1, \dots, x_n)$ the data points.

4. CONVERGENCE OF THE STOCHASTIC APPROXIMATION ALGORITHM

Theorem 2: Let the sequence $\{a_k\}_{k\geq 0}$ of positive numbers satisfies the conditions:

(a) $a_0 = 1,\ 0 < a_k < 1 \quad \forall k > 1$

(b) $\sum_{k=0}^{\infty} a_k = \infty$

(c) $\sum_{k=0}^{\infty} (a_k)^2 < \infty$

Given that $\{d^k\}$ is a sequence of random vectors satisfying (15) and (16), and then the stochastic sequence $\{x^k\}_{k\geq 0}$ in H defined iteratively from $x_0 \in D(f)$ by $x^{k+1} = x^k - a_k d^k$ converges strongly to a unique solution $\{x^* : \nabla f(x^*) = 0\} \in H$ almost surely.

Proof: Let $\Lambda_k = a_k \|d^k - \nabla f(x^k)\|^2 = \frac{\sigma^2}{N}, 0 < \sigma^2 < \infty$, then $\{\Lambda_k\}$ is a sequence of independent random variables. From (15), $E\Lambda_k = 0\ for\ each\ k,$ thus the sequence of partial sums $S_k = \sum_{i=1}^{k} \Lambda_i$ is a Martingale.

But $E{S_k}^2 = \sum_{i=1}^{k} E{\Lambda_i}^2 = \sum_{i=1}^{k} a_i^2 E\left\|d^i - \nabla f(x^i)\right\|^2 = \frac{\sigma^2}{N} \sum_{i=1}^{k} a_i^2$.

The convergence of the series $\sum a_i^2$ in (Theorem 2(c)), implies $\sum_{i=1}^{\infty} E\Lambda_i^2 < \infty$. Thus by Martingale convergence theorem (see [10]), we have

$\sum_{k=1}^{\infty} \Lambda_k < \infty$, so that

$\lim_{k\to\infty} a^k \|d^k - \nabla f(x^k)\| = 0 \implies d^k \to \nabla f(x^k)$.

It is obvious that the stochastic analog of (15) under suitable hypotheses, behaves asymptotically as (15), and it implies the convergence properties of (15) are preserved when ∇f is replaced by

d^k. It follows that the stochastic iteration scheme: $x^{k+1} = x^k - a^k d^k$ converges strongly to unique solution $\{x^*: \nabla f(x^*) = 0\} \in H$.

From (18) d^k strongly approximates $\nabla f(x^k)\, at\, x^k$. Furthermore, it can be established that $\|d^k - \nabla f(x^k)\| \leq \|d_1^k - \nabla f(x^k)\|$ for all least-square approximations. Then the stochastic approximation algorithm scheme generated by the stochastic sequence $\{d^k\}$ is then given as follows:

Let x^k be an estimate of x^*

(a) Compute $d^k \approx \nabla f(x^k)$

(b) Compute a_k

(c) Compute $x^{k+1} = x^k - a_k d^k$

(d) Check for convergence:

Is $\|x^{k+1} - x^k\| < \lambda, \lambda > 0$?

Yes: $x^{k+1} = x^k$

No: Set $k = k + 1$ and return to (a)

5. CONCLUSION

We have examined a stochastic iteration method that converges strongly to the solution of variational inequalities problems in Hilbert space. This approach is similar to that in the finite dimensional case, but developed for the problems in the Hilbert space setting.

The inequality problem is reformulated to a root finding problem of determining the zero of a monotone operator in the Hilbert space.

However, the classical deterministic methods for approximating the zeros of monotone operators are effective for a large problem sets. The stochastic method is able to handle many of the problems for which deterministic method are inappropriate.

Further research area and possible extension is the regularity of existence of the solution. The regularity solution is of great importance to crucial part in the derivation of the error estimate for the finite-element approximations of variational inequalities. The regularity problem for the general and quasi variational inequalities is still an area for future work.

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